Chapter 4

Financial Operations and their Evaluation: Decisional Criteria

4.1. Calculation of capital values: fairness

In financial practice there is often the problem of evaluation, at a given time and based on a given exchange law, of a finite set of financial supplies, i.e. of incomes or payments to be made at a fixed time. It is easy to generalize about an infinite number of supplies or to a continuous flow of payments as a theoretical model which approximates a sequence of financial transactions of small amount with close maturity.

Such a set is called a *financial operation* because it is the financial reflex of economic acts regarding flows or funds (like transferring assets, payments for services, loans with a unique or periodic repayment schedule, installation or management of industrial equipment, etc.).

Referring to the concept of *financial supply (T,S)* as well as to the *equivalence principle* based on a given exchange law, a *financial operation O*, which we will firstly consider as discrete payments, can then be defined as *union of supplies*, using

$$O = \bigcup_{h=1}^{n} (T_h, S_h) \tag{4.1}$$

If $n \to \infty$, it is necessary to introduce some conditions. Without loss of generality we will consider $\{T_h\}$ increasing with h, i.e. in chronological order; in addition, $S_h>0$ are incomes for the agent “A” whereas $S_h<0$ are payments. The operation O is also called a *financial project* if it is referred to dated amounts that are expected by a feasible project.

The operation O can be alternatively expressed in transposed form with respect to (4.1), using a pair of n-dimensional vectors ("maturities $\{T_h\}$, cash flow[1] $\{S_h\}$", h=1,...,n), instead of a n-tuple of two-dimensional vectors which identify the supplies "time T_h, amount S_h". Therefore, we can write

$$O = (T_1, T_2,...,T_n)\&(S_1, S_2,...,S_n) = \{T_h\}\&\{S_h\} \qquad (4.1')$$

where & = *correspondence between vector components* and where corresponding pairs (T_h,S_h) with the same h identify the supply.

It is usual to distinguish in (4.1) between the cases n=2, which give rise to a *simple operation*, and n>2, which give rise to a *complex operation*. A simple operation, if S_1 and S_2 have opposite sign, is just an exchange between two amounts with different maturities.

Let us consider an economic agent "A" who wants to value O at time T, based on an indifference relation ≈ which gives rise to the exchange factors $z(X,Y)$, given by (2,5'), which express the used financial law. We then define as *capital value* (or just *value*) of the operation O at time T (from the point of view of agent "A") the amount $V(T;O,z)$ so that "A" considers O to be fairly exchangeable with the supply $(T, V(T;O,z))$. In other words, from the point of view of "A" there is indifference between obtaining the supplies O and acquiring the amount $V(T;O,z)$ in T.

We can apply what we have said above to calculate the selling value of a company. If O, expressed by (4.1), concerns all financial transactions related to its management and expected by one party (for example, the seller) in $T<T_1$ (and then $T<T_h, \forall h$), then $V(T;O,z)$ is the value in T given to the company based on the law z, which has to be compared with the offered price to judge whether it is convenient to sell[2].

To measure $V(T;O,z)$ we can consider that, because of results in section 2.2, the amount exchangeable in T with S_h in T_h is $S'_h = S_h{:}z(T_h,T)$, and this is the value in T of (T_h,S_h); furthermore, we will usually assume the *additive property*, by which the

1 Correctly speaking, flow should be used for the continuous case, but it is also frequently used in finance for the discrete case when there is a sequence of payments.

2 The comparison between values (subjective, as a consequence of the choice of z) and prices (objective, because they are fixed by the market) is the basis of the decisions and choices theory between "financial projects" that we will consider later.

value in T of the union of supplies $O = \bigcup_{h=1}^{n}(T_h, S_h)$ is the sum of the values at the same time T of each supply. Therefore

$$V(T;O,z) = \sum_{h=1}^{n} S'_h = \sum_{h=1}^{n} S_h z(T_h, T)\,. \tag{4.2}$$

When there is no ambiguity on O and on the law z, we will write $V(T)$ instead of $V(T;O,z)$.

We will say that "A" considers *fair* (or *well-balanced*) the operation O in T based on his choice of law z or, briefly, that *the operation O is fair in T if and only if* $V(T;O,z) = 0$.

Adopting exchange laws $z(X,Y)$ which are always positive with any X and Y, the *fairness of O* implies that there is no concordance in algebraic sign of all amounts of O.

A simple fair operation is a pure exchange (i.e. a *repurchase agreement*) balanced according to z, resulting in $S_2 = -S_1 \cdot z(T_1,T_2)$.

Complements on fair operations

We will say that an exchange law identified by ≈, according to which the fairness of an operation O is valued, verifies the *invariance property* if an operation considered fair in T_0 is also fair in all other times T. Furthermore, given that fairness implies the zero value of O, the additive property implies that the union of two or more operations (defined as the union of the sets of their supplies), all judged fair in T_0, is fair in T_0.

As invariance does not generally hold, then the value $V(T;O,z)$ if it is zero in $T=T_0$ can become different from zero in a different T; it is then necessary to specify the evaluation time. However, given that, as can be proved, the strong decomposability implies additivity and invariance together, if z satisfies such a property, the judgment of fairness of the operations does not depend on the evaluation time T.

It is important to observe here that – given that an exchange law implies the payment of interest for the deferring of the availability of a principal amount – saying that "A" considers the operation O to be fair having assumed the law $z(X,Y)$ means that O gives exactly the return expressed by z. In other words, i.e. with inflows and outflows at the times T_h on a profitable account ruled by such a law, if $V(T_0;O,z) = 0$, this means that in T_0 the evaluations of S_h, taking into account the interest, are balanced. If instead $V(T_0;O,z) > 0$ (< 0), the operation O gives rise to a spread of positive (negative) returns added to the return implied by the law *z*. This is the starting point of the theory of comparisons and choices between financial operations on the basis of the returns.

If the law *z* identified by ≈ is *uniform, not decomposable*, and therefore the exchange factor has the form $g(\tau)$ (see (2.40)), it is enough to replace in (4.2) $z(T_h,T) = g(\tau_h)$, where $\tau_h = T\text{-}T_h$, and then

$$V(T;O,z) = \sum_{h=1}^{n} S_h \; g(T - T_h) \tag{4.3}$$

Particular cases

a) Simple delayed interest (SDI) law at rate i and its conjugate rational discount (rd)

$$\text{SDI: } g(\tau) = 1 + i\,\tau, \text{ if } \tau > 0; \text{ rd: } g(\tau) = 1/(1 + i\,|\tau|), \text{ if } \tau < 0$$

b) Simple advance interest (SAI) law at rate d and its conjugate simple discount (sd)

$$\text{SAI: } g(\tau) = 1/(1 - d\,\tau), \text{ if } \tau > 0; \text{ sd: } g(\tau) = 1 - d\,|\tau|, \text{ if } \tau < 0$$

If the law *z* identified by ≈ is *strongly decomposable (s.dec), non-uniform,* characterized by an intensity $\delta(\lambda)$ as a function of current time λ (see section 2.4), the exchange factor in (4.2) is written[3]: $z(T_h,T) = \exp\left(\int_{T_h}^{T} \delta(\lambda)\, d\lambda\right)$.

If the law *z* identified by ≈ is *uniform and also s.dec*, it falls within (as shown in Chapter 2) the *exponential regime* where ≈ is a relation of uniform equivalence. Therefore, it gives rise to the following case.

c) Continuously compound interest (CCI) law with intensity δ and its conjugate continuously compound discount (CCD)

$$\text{CCI: } g(\tau) = e^{\delta\tau}, (\tau > 0); \text{ CCD: } g(\tau) = e^{\delta\tau}, (\tau < 0)$$

and then

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3 It is well known that $\exp\left(\int_a^b \delta(\lambda)\, d\lambda\right)$ is often used, for simplicity, instead of $e^{\int_a^b \delta(\lambda)\, d\lambda}$.

$$V(T;O,z)=\sum_{h=1}^{n} S_h\ e^{\delta(T\text{-}T_h)} = \sum_{h=1}^{n} S_h\ (1+i)^{T\text{-}T_h} \tag{4.4}$$

Exercise 4.1

Let us consider the following operation

O = {(0, -1,500)∪(2.5, -1,850)∪(3.5, 520∪(5, 4,500)}

where time is measured in months, and let us calculate its value in T=4 using the SDI law with an annual rate i = 5.5% and its conjugate rd.

A. By applying (4.3), we obtain

$$V(4)=-1{,}500\ (1+0.055\frac{4}{12})-1{,}850\ (1+V(4)=0.055\frac{1.5}{12})+520\ (1+0.055\frac{0.5}{12})$$
$$+\ 4{,}500\ \frac{1}{1+0.055\frac{1}{12}}\ =1{,}610.44$$

Exercise 4.2

Let us consider the same operation as in Exercise 4.1, i.e.

O = {(0, -1,500)∪(2.5, -1,850)∪(3.5, 520)∪(5, 4,500)}

where time is measured in months, and let us calculate its value at time T=4 using the SAI law with an annual discount rate d equivalent to i =5.5% and its conjugate sd.

A. The equivalent rate d is 0.052133. Applying (4.3)

$$V\left(4\right)=-1{,}500\,\frac{1}{1\text{-}0.052133\,\frac{4}{12}}-1{,}850\,\frac{1}{1\text{-}0.052133\,\frac{1.5}{12}}+$$
$$520\,\frac{1}{1\text{-}0.052133\,\frac{0.5}{12}}+4{,}500\,(1-0.052133\,\frac{1}{12})=1{,}612.920$$

Exercise 4.3

Let us consider the same operation as in Exercise 4.1, i.e.

$$O = \{(0, -1{,}500)\cup(2.5, -1{,}850)\cup(3.5, 520)\cup(5, 4{,}500)\}$$

where time is measured in years, and let us calculate its value at time T=4 using the exponential exchange law with an annual interest i = 6%.

A. The equivalent instantaneous intensity δ is 0.058269. By applying (4.4)

$$g(\tau) = e^{0.058269\,\tau} = 1.06^{\tau}$$

and then

$$V(4) = -1500\cdot 1.06^{4} - 1850\cdot 1.06^{1,5} + 520\cdot 1.06^{0,5} + 4500\cdot 1.06^{-1} = 867.97$$

The value can also be found with an Excel spreadsheet where in the 1st row we put the rate values and evaluation time, in the 2nd row the column's titles and from the 3rd to 6th rows the needed values: terms and amounts of supplies; exchange factors from the terms to 4; amount valued at time 4, then the sum gives $V(4)$ = 867.97. The following table is obtained.

Rate = 0.06		Time = 4	
Term	*Amount*	*Exchange factor at 4*	*Value at 4*
0.0	–1,500.00	1.2624770	–1,893.715
2.5	–1,850.00	1.0913368	–2,018.973
3.5	520.00	1.0295630	535.373
5.0	4,500.00	0.9433962	4,245.283
		V(4) =	867.967

Table 4.1. *Calculation of values*

The Excel instructions are as follows. B1: 0.06; D1: 4. The first two rows are for data and column titles; from the 3rd to 6th rows:

– column A (maturity): A3: 0; A4: 2.5; A5: 3.5; A6: 5;

– column B (amounts): B3: –1,500; B4: –1,850; B5: 520; B6: 4,500;

– column C (exchange factors in 4): C3: = (1+B$1)^(D$1-A3); copy C3, then paste on C4 to C6;

– column D (evaluation in 4): D3: = B3*C3; copy D3, then paste on D4 to D6; (value at time 4): D7: = SUM(D3;D6).

Financial operation with continuous flow

Let us consider briefly the calculation of the value of operations with continuous flow. In the continuous case the elementary supply is expressed by $[t, \varphi(t)dt]$, with $\varphi(t)$ defined in $t' \leq t \leq t''$ and therein continuous (however, to calculate the values the integrability is enough). Then (4.2), (4.3), (4.4) become respectively

$$V(T;O,z) = \int_{t'}^{t''} \varphi(t) z(t,T)\, dt \tag{4.2'}$$

$$V(T;O) = \int_{t'}^{t''} \varphi(t) g(T-t)\, dt \tag{4.3'}$$

$$V(T;O) = \int_{t'}^{t''} \varphi(t) e^{\delta(T-t)}\, dt \tag{4.4'}$$

Using an s.dec and non-uniform exchange law, the exponential in (4.4') becomes $\exp\left(\int_t^T \delta(\lambda) d\lambda\right)$.

Naturally, there can be a *mixed operation*, which puts together continuous and discrete operations, and due to additivity the value will be given by the sum of the values (4.2) and (4.2') (or of the values for the other particular cases).

4.2. Retrospective and prospective reserve

With reference to an operation O and an exchange law $z(X,Y)$, let us assume T in the interval $[T_1,T_n]$ is *logically distinct*[4] from each T_h. Let r be the number of supplies before T and n-r those after T $(0 \leq r \leq n)$.

We then define as *retrospective reserve* (briefly: *retro-reserve*) of O at time T according to z the amount $M(T;O,z)$ given by the opposite of the value in T of the sub-operation O' consisting of the set of all supplies of O *before T*. If there is no ambiguity, we write $M(T)$ instead of $M(T;O,z)$.

We define as *prospective reserve* (briefly: *pro-reserve*), or *residual value*, of O at time T according to z the amount $W(T;O,z)$ given by the value in T of the sub-operation O'' consisting of the set of all supplies of O *after T*. If there is no ambiguity, we write $W(T)$ instead of $W(T;O,z)$.

4 By *logically distinct* we mean that T is different from the time of payment T_h or, if they coincide, that there is a method to establish if the corresponding S_h has to be added to the payments before or after T (for example, with a rule of “delayed” or “advance” payment per period).

Because of (4.1) we have:

$$M(T;O,z) = -\sum_{h=1}^{r} S_h \, z(T_h,T) \tag{4.5}$$

$$W(T;O,z) = \sum_{h=r+1}^{n} S_h \, z(T_h,T) \tag{4.6}$$

Notice that in (4.5) we have only accumulation processes and in (4.6) only discount processes. Therefore, due to (2.5'), instead of $z(T_h,T)$ we can use $m(T_h,T)$ in (4.5) and $a(T_h,T)$ in (4.6).

If the exchange law is *uniform (non-decomposable),* (4.5) and (4.6) can be written, giving $t_h = |\tau_h|$ and recalling (2.42)

$$M(T) = -\sum_{h=1}^{r} S_h \, u(t_h) \tag{4.7}$$

$$W(T) = \sum_{h=r+1}^{n} S_h \, v(t_h) \tag{4.8}$$

If the exchange law is *s.dec (non-uniform)* with intensity $\delta(\lambda)>0$, the exchange factor in (4.5) and (4.6) is written as: $z(T_h,T) = \exp\left(\int_{T_h}^{T} \delta(\lambda)\, d\lambda\right)$.

In particular, if the exchange law is *exponential* with rate *i*, the expressions for the reserves are

$$M(T) = -\sum_{h=1}^{r} S_h \, (1+i)^{(T-T_h)} \; ; \; W(T) = \sum_{h=r+1}^{n} S_h \, (1+i)^{-(T_h-T)} \tag{4.9}$$

From the definitions, whichever exchange law is used for the operation *O*, it follows that

$$V(T) = W(T) - M(T), \quad \forall T \tag{4.10}$$

Therefore, if the exchange law implies fairness for *O* at time T_0 *only for this time* it follows that

$$M(T_0) = W(T_0) \tag{4.11}$$

However, if for such a law the invariance property holds (in particular if the s.dec holds) (4.11) implies that

$$M(T) = W(T), \quad \forall T \tag{4.12}$$

In the *continuous case*, with operations spread in *(t',t")*, the reserves are obtained by adopting the previous formulae. Therefore:

– in the general case of two variables law *z*:

$$M(T;O,z) = -\int_{t'}^{T} \varphi(t) z(t,T) dt \;\; ; \;\; W(T;O,z) = \int_{T}^{t''} \varphi(t) z(t,T) dt \tag{4.5'}$$

– with the non-decomposable uniform law:

$$M(T) = -\int_{t'}^{T} \varphi(t) g(T-t) dt \;\; ; \;\; W(T) = \int_{T}^{t''} \varphi(t) g(T-t) dt \tag{4.7'}$$

– with the exponential law:

$$M(T) = -\int_{t'}^{T} \varphi(t) e^{\delta(T-t)} dt \;\; ; \;\; W(T) = \int_{T}^{t''} \varphi(t) e^{-\delta(t-T)} dt \tag{4.9'}$$

while with any s.dec law with intensity $\delta(\lambda)$ the exponentials in (4.9') must be replaced by $\exp(\int_{t}^{T} \delta(\lambda)\, d\lambda)$.

With a *mixed operation*, valued with a law of two variables, the retro-reserve is obtained by adding *M(T)* written in (4.5) and (4.5') and the pro-reserve by adding *W(T)* written in (4.6) and (4.5'); when particular regimes are used, the aforementioned corresponding expressions for *M(T)* and *W(T)* must be added.

Observation

The previous definitions need some interpretation. As mentioned in footnote 1 of Chapter 1, a financial transaction is usually coupled with a real transaction of opposite side. In particular, an operation *O* can be concerned with financial transactions, managed by Mr A, connected with the management of a company for the production or the trade of assets or services, which we call *project O*. According to the accounting principle of “double entry”, such transactions are registered by Mr A on an account giving interest, assigned to *O*; each payment implies a charging of the account and then the creating of a credit of Mr A (or the settlement of a debt) while each income implies a crediting on the account and then the creating of a debt of Mr A (or the settlement of a credit).

Given that according to the definitions the retrospective reserve *M(T;O,z)* represents the financial statement of Mr A at time *T* following the transactions with the sub-operation *O'*; it is positive or negative (i.e. a credit or a debit for Mr A) depending on the fact that before *T* is greater the number of payments or else incomes for Mr A, valued financially in *T* through the exchange law *z*. Consequently, *M(T;O,z)* is the amount that, if the supplies of *O* subsequent to *T* would cancel, Mr A should cash (algebraically) in *T* such that the resulting operation, consisting of {*O'*∪[*T*, *M (T;O,z)*]}, would be fair in *T*.

Vice versa, the prospective reserve *W(T)* is the capital value in *T* of the supplies of the sub-operation *O"*, i.e. the amount that, if the supplies of *O* before *T* would cancel, Mr A should pay (algebraically) in *T* such that the resulting operation, consisting of {*O"*∪[*T*, -*W(T;O,z)*]}, is fair in *T*.

The names *retrospective reserve* and *prospective reserve* are also used (referring to expected values) in the stochastic financial insurance operations.

Exercise 4.4

Let us consider again Exercise 4.3 and observe that *O* is not fair, given that *V*(4)=867.97. It is enough to add the supply (4, -867.97) to obtain, using a rate of 6%, a fair operation (at each time, given that the adopted law is decomposable)

$$\hat{O} =\{(0, -1,500)\cup(2.5, -1,850)\cup(3.5, 520)\cup((4, -867.97)\cup(5; 4,500)\}$$

Calculate the reserves of $\hat{O}$ in *T*=3 verifying the validity of (4.12).

A. By applying (4.9) we obtain for $\hat{O}$:

$$M(3) = 1,500\cdot 1.06^{3} + 1,850\cdot 1.06^{0.5} = 3,691.22$$

$$W(3) = -867.966\cdot 1.06^{-1} + 520\cdot 1.06^{-0.5} + 4,500\cdot 1.06^{-2} = 3,691.22$$

Using an Excel spreadsheet for the same calculation, we have to proceed as follows. For the calculation of the retro-reserve it is necessary to take into account only the supplies before *T*=3; therefore, expanding along the columns, the supplies below are not considered; vice versa for the calculation of the pro-reserve it is necessary to take into account only the supplies after *T*=3; therefore, expanding along the columns, the supplies above are not considered. To do this (if the Excel macros are not applied), given that

$$a' = (a+|a|)/2a = 1 \text{ if } a>0, = 0 \text{ if } a<0; \; a'' = (a-|a|)/2a = 0 \text{ if } a>0, = 1 \text{ if } a<0$$

then for the calculation of *M(T)* in the 1st of (4.9) $-S_h(1+i)^{(T-T_h)}$ is preserved if $T-T_h > 0$ and we use 0 if $T-T_h < 0$; on the contrary for the calculation of *W(T)* in the 2nd formula of (4.9) we use 0 if $T-T_h > 0$ and $S_h(1+i)^{-(T_h-T)}$ is preserved if $T-T_h < 0$. Therefore, setting such values by columns, we obtain the retro-reserve adding the amounts valued at time 3 multiplied by a', while the pro-reserve is obtained adding the amounts valued at time 3 multiplied by a''.

i = 0.06				T = 3			
T_h	S_h	T-T_h	$S_h(T)$	a'	a''	$Am.(T>T_h)$	$Am.(T<T_h)$
0.0	-1,500.000	3.00	-1,786.52	1.0	0.0	1,786.524	0.000
2.5	-1,850.000	0.50	-1,l904.69	1.0	0.0	1,904.692	0.000
3.5	520.000	-0.50	505.07	0.0	1.0	0.000	505.069
4.0	-867.967	-1.00	-818.84	0.0	1.0	0.000	-818.837
5.0	4,500.000	-2.00	4,004.98	0.0	1.0	0.000	4,004.984
						3,691.216	3,691.216

Table 4.2. *Calculation of retro-reserves and pro-reserves*

The Excel instructions are as follows. The first two rows are for data and titles; C1: 0.06; F1: 3. From the 3rd to 7th rows:

– column A (maturity): A3: 0; A4: 2.5; A5: 3.5; A6: 4; A7: 5;

– column B (amounts): B3: –1,500; B4: –1,850; B5: 520; B6: 867.967; B7: 4,500;

– column C (maturity): C3: = F$1-A3; copy C3, then paste on C4 to C7;

– column D (amounts valued in 3): D3:= B3*(1-C$1)^(F$1-A3); copy D3, then paste on D4 to D7;

– column E (a' = indicates *M(T)*): E3: = (C3+ABS(C3))/2/C3; copy E3, then paste on E4 to E7;

– column F (a'' = indicates *W(T)*): F3: = (C3-ABS(C3))/2/C3; copy F3, then paste on F4 to F7;

– column G (amounts for *M(T)*): G3: = -D3*E3; copy G3, then paste on G4 to G7;

– column H (amounts for *W(T)*): H3: = D3*F3; copy H3, then paste on H4 to H7; (retro-reserve in 3): G8: = SUM(G3;G7); (pro-reserve in 3): H8: = SUM(H3;H7). Then: G8 = H8.

Exercise 4.5

From the data of the operation considered in Exercise 4.1, we observe that

$$\hat{O}=\{(0,-1{,}500)\cup(\tfrac{2.5}{12},-1{,}850)\cup(\tfrac{3.5}{12},520)\cup(\tfrac{4}{12},-1{,}610.44)\cup(\tfrac{5}{12},4{,}500)\}$$

is fair in $T_0=\frac{4}{12}$. Verify the validity of (4.11) if $T_0=\frac{4}{12}$ and its non-validity (i.e. unfairness of $\hat{O}$) if $T_0=\frac{1}{12}$, due to the non-decomposability of the adopted laws.

A. We add $(\frac{4}{12}, -1{,}610.44)$ to the payments after $\frac{4}{12}$. By calculating the accumulation with an SDI law at the annual rate of 5.5% we obtain $M(\frac{4}{12})$ = 3,390.18; by calculating the discount with an RD law with the same rate we obtain $V(\frac{4}{12})$ = 3,390.18 = $M(\frac{4}{12})$. By evaluating at the time $\frac{1}{12}$, we obtain $M(\frac{1}{12})$ = 1,506.87; instead $V(\frac{1}{12})$ = 1,507.12 $\neq M(\frac{1}{12})$, then $\hat{O}$ is not fair if valued at such time.

The "differential equation of accumulated value" with principal flow

A particular mixed O in the interval $(0,T)$, which at the same time allows a generalization of (3.35), is obtained by considering an initial supply $(0,S_0)$ and other later supplies with infinitesimal amounts $(t,\varphi(t)dt)$, $(0 \leq t \leq T)$, following a continuous principal flow $\varphi(t)$. This is useful to schematize the management of a small firm, by considering an initial cost for establishment and then small financial transactions as inflows and outflows.

The accumulation of interest always proceeds according to the cci law with instantaneous intensity δ. With such a hypothesis the retro-reserve $M(t)$ varies for effect of the financial transactions due to the flow $\varphi(t)$, as well as for the continuous accumulation of interest, due in S_0 and in $\varphi(\tau)d\tau$, $(\tau \leq t)$ according to the flow $\delta\cdot M(t)$. This process can then be obtained by solving the following linear and non-homogenous differential equation:

$$M'(t) = \delta\, M(t) - \varphi(t) \tag{4.13}$$

In fact, generalizing (3.34'), $\forall t \in (0,T)$ in the given hypothesis the dynamics of the retro-reserve are described by

$$M(t+dt) = M(t) + \delta\, M(t)\, dt - \varphi(t)dt + o(dt) \tag{4.13'}$$

Dividing by dt and taking the limit $dt\to 0$, given that $o(dt)/dt\to 0$, (4.13) follows. This equation is also called the *differential equation of the accumulated value*. It is indeed easy to see that the retro-reserve in T coincides with the amount, valued in T, of the invested principal due to the outflows, subtracting the inflows, before T.

The analytical solution of (4.13) is immediate. In fact, by multiplying both members by $e^{-\delta t}$, it is soon found that the general integral is

$$M(t) = e^{\delta t}\left\{-\int \varphi(t)e^{-\delta t}dt + \text{constant}\right\} \qquad (4.14)$$

and, due to the continuity of *M(t)* after 0, the particular solution of (4.13), where $M(0) = -S_0$, is

$$M(t) = e^{\delta t}\left\{M(0) - \int_0^t \varphi(\tau)e^{-\delta\tau}d\tau\right\} = M(0)e^{\delta t} - \int_0^t \varphi(\tau)e^{\delta(t-\tau)}d\tau \qquad (4.14')$$

Value (4.14') can be financially interpreted observing that, due to the s.dec of the exponential exchange law, the retro-reserve in t can be obtained by accumulating in t the property evaluations in 0 connected to the financial transactions occurring between 0 and t (and then of opposite sign).

If an s.dec law is used with intensity $\delta(\lambda)$ instead of the cci, solution (4.14') is generalized as

$$M(t) = M(0)\exp\left(\int_0^t \delta(\lambda)d\lambda\right) - \int_0^t \varphi(\tau)\exp\left(\int_\tau^t \delta(\lambda)d\lambda\right)d\tau \qquad (4.15)$$

EXAMPLE 4.1.– Mr. B opens in a financial institution a c/a both for deposit (when Mr B is in credit) and for lending (when Mr B is in debt), ruled by an exponential exchange regime and with *reciprocal rate*, i.e. with the same instantaneous intensity, both the earned interest on the credits and the passive interest on the debts are obtained and converted time by time. Assuming the monetary unit MU = €1,000 euros, let us suppose that the transaction in the c/a in the interval $(0,t)$ is given by a deposit in 0 of MU 25.48 followed by deposits and withdrawals based on a continuous flow which is assumed with a parabolic shape $\varphi(\tau) = a+b\tau+c\tau^2$, $(0\leq\tau\leq t)$. Let us use t=2, finding the function $\varphi(\tau)$ by interpolation on the basis of the values at times 0, 1, 2, that are respectively: $\varphi(0) = -4$ (= infinitesimal payment $-4dt$), $\varphi(1) = +5$ (= infinitesimal income $5dt$); $\varphi(2) = +12$ (= infinitesimal income $12dt$). Mr B wants to estimate the retro-reserve at time 2, i.e. his position $M(2)$ (of credit if $M(2)>0$, of debit if $M(2)<0$), with a CCI law at the reciprocal annual rate of 5%.

To do this we firstly need to calculate the flow function imposing its passage for the points $P_i = (i, \varphi(i))$, i=1,2,3, and deducing the parameters a, b, c. Then we have to solve the linear system

$$a = -4\ ;\ a+b+c = 5;\ a+2b+4c = 12$$

which has as unique solution: a = -4 , b = +10 , c = -1. Then: $\varphi(\tau) = -4+10\tau-\tau^2$. In addition, $\delta = \ln(1.05) = 0.04879$. Therefore, due to (4.14'), $M(2)$ is obtained from the following expression

$$M(2) = e^{0.09758}\left\{25.48 - \int_0^2 [-4+10\tau-\tau^2]e^{-0.04879\tau}d\tau\right\}.$$

Integrating *by parts*: $\int [-4+10\tau-\tau^2]e^{-0.04879\tau}d\tau =$

$$= \frac{-e^{-0.04879\tau}}{0.04879}\left[\left\{-4+\frac{1}{0.04879}\left(10-\frac{2}{0.04879}\right)\right\}+\left(10-\frac{2}{0.04879}\right)\tau-\tau^2\right]+cost.$$

and then the integral: $\int_0^2 [-4+10\tau-\tau^2]e^{-0.09758}d\tau =$

$$= \frac{1-e^{-0.09758}}{0.04879}\left\{-4+\frac{1}{0.04879}\left(10-\frac{2}{0.04879}\right)\right\}-\frac{e^{-0.09758}}{0.04879}\left\{\left(10-\frac{2}{0.04879}\right)2-4\right\} =$$
$$= 8.64434$$

Therefore: $M(2) = e^{0.09758}\{25.48-8.64434\} = 18.56131$ MU

It follows that, withdrawing from $\varphi(\tau)$ the accumulated incomes on the accumulated payments, the result in t=2 is negative, i.e. there is a decrement of credit, equal to €9,530.38, compared to 25480·1.1025 = €28,091.69 = credit in the c/a that Mr B would have in absence of the flow $\varphi(\tau)$ in the interval [0,2].

4.3. Usufruct and bare ownership in "discrete" and "continuous" cases

Assuming an s.dec exchange law z and supposing that the operation O is fair at a given time T_0 and then, due to the invariance of z, $\forall T$ (if this is not true, to make O fair it is enough to add to the original supplies $(T_0,-V(T_0))$), it is important, under the practical point of view the *decomposition into two parts of the pro-reserve W(T)* at time T (where inside the symbol in $W()$ are implicit the symbols O and z):

a) the first, called *usufruct* and indicated by *U(T)*, is the evaluation in T of the financial transactions due only to the interest settled after T;

b) the second, called *bare ownership* and indicated by *P(T)*, is the evaluation in T of the remaining transactions, i.e. the supplies of O'' without interest.

Then by definition

$$W(T) = U(T) + P(T), \ \forall T \tag{4.16}$$

and, once *W(T)* and *U(T)* are calculated, the bare ownership is given by the difference. The distinction made in (4.16) is important, because in a financial operation there can be difference in the owners of the rights to the two supplies.

The calculation of the usufruct and the bare ownership is usually done according to the *discrete* scheme that approximates the scheme of the interest continuous formation. Taking as an example a lending operation with periodic payments, at such times the *interest shares* are also calculated based on the current debt position and the time that has passed since the last payment. The *usufruct in discrete case U(T)* is then the present value in *T* of the interest shares after *T* and the *bare ownership in discrete case P(T)* is the present value in *T* of the principal share after *T*. On this point we will talk about amortizations and their evaluation in the following chapters.

However, usufruct and bare ownership can be calculated rigorously according to the *continuous* scheme, assuming the *continuous payment of interest* as obtained. Indicating by $\tilde{U}(t)$ and $\tilde{P}(t)$ the *usufruct and bare ownership in continuous case*, we have

$$W(t) = \tilde{U}(t) + \tilde{P}(t), \forall t \tag{4.16'}$$

For $\tilde{U}(t)$ and $\tilde{P}(t)$ clearly additivity holds, i.e. they are obtained as the sum of the usufructs and the bare ownerships, calculated by the continuous scheme, of each supply of *O"*.

Let us find the expression for $\tilde{U}(t)$ and $\tilde{P}(t)$ in the hypothesis that *O* consists only of transactions (T_h,S_h). Let $\delta(\lambda)$ be the instantaneous intensity connected to an s.dec law and then dependent only on the current time λ). Also let T_h $(h = r+1,...,n)$ be the maturity times later than *T*. We obtain

$$\tilde{U}(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\} \int_T^{T_h} \delta(\lambda)d\lambda \tag{4.17}$$

$$\tilde{P}(t) = W(t) - \tilde{U}(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}\left[1 - \int_T^{T_h} \delta(\lambda)d\lambda\right] \tag{4.18}$$

Proof

Given the additivity, we first consider one supply (T_h,S_h) with *h* fixed between $r+1$ and *n*. The earned interest between time *X* and *X+dX* is (not considering errors that vanish with the following integration) $M(X)\delta(X)dX$, but because *z* is s.dec and *O* is fair, we have $M(X) = W(X)$, $\forall X$. To calculate $\tilde{U}(t)$ we must integrate from *T* onwards the interest $M(X)\delta(X)dX = W(X)\delta(X)dX$ discounted in T, multiplying by $\exp\left\{-\int_X^T \delta(\lambda)d\lambda\right\}$. The prospective reserve on (T_h, S_h) is: $W_h(X) = S_h \exp\left\{-\int_X^{T_h} \delta(\lambda)d\lambda\right\}$ if $X \le T_h$; otherwise $W_h(X) = 0$ and then the integration of interest for $X > T_h$ gives no contribution. After some calculations we obtain:

$$\tilde{U}(t) = \sum_{h=r+1}^{n} W_h(X)\delta(X)\exp\left\{-\int_T^X \delta(\lambda)d\lambda\right\}dX =$$
$$\sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}\int_T^{T_h} \delta(\lambda)d\lambda$$

i.e. (4.17) holds. Expression (4.18) for $P(t)$ is obtained subtracting the 2nd member of (4.17) from $W(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}$.

Let us find the expression for $\tilde{U}(t)$ and $\tilde{P}(t)$ maintaining the hypothesis of fair O, which we assume to be mixed, extended to the time interval (t', t'') with discrete and continuous supplies, and assuming an exponential exchange law with intensity δ. Because of uniformity of the law, we can assume the beginning of the operation in the origin of time, i.e. $t' = 0$. Using (4.17) and (4.18) for the discrete component and considering that for the continuous component it is enough that the sums of the amounts S_h, with $t_h \in (t, t'')$, are replaced by integral on the time of the flows $\varphi(\lambda)$ with $\lambda \in (t, t'')$, the following formulae are easily obtained

$$\tilde{U}(t) = \delta \sum_{h=r+1}^{n} S_h e^{-\delta(t_h - t)}(t_h - t) + \delta \int_t^{t''} \varphi(\lambda) e^{-\delta(\lambda - t)}(\lambda - t)d\lambda \qquad (4.17')$$

$$\tilde{P}(t) = \sum_{h=r+1}^{n} S_h e^{-\delta(t_h - t)}\left[1 - \delta(t_h - t)\right] + \int_t^{t''} \varphi(\lambda) e^{-\delta(\lambda - t)}\left[1 - \delta(\lambda - t)\right]d\lambda \qquad (4.18')$$

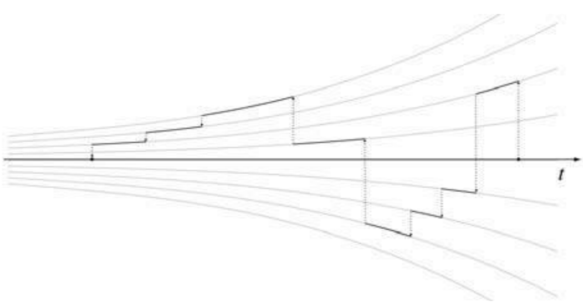

Figure 4.1. *Plot of the values M(t) =W(t) if O is fair*

4.4. Methods and models for financial decisions and choices

4.4.1. *Internal rate as return index*

We will now discuss the parameters of implicit return in a financial operation, for which we have already considered the evaluation of the whole or of some of the parts (reserve, usufruct or bare ownership), as well as the decisional criteria for financial operations (discrete) $O = \{T_h\}\&\{S_h\}$ that, considering the set of economic and technical facts below the set of supplies (S_h,T_h), we will call *financial projects* (of whatever type: realized in agriculture, industry, commerce, services, etc.). It is fundamental to give a general definition of *internal rate* (of return for the investor, of costs for the borrower), relative to O.

Let us recall the advantage of using a uniform financial law, according to which all evaluations can be performed at time zero, and then the exchange factors are all discount factors. If the law is also decomposable, which we will suppose to be true from now on, then it is the exponential law, characterized by a constant intensity. With reference to the payments, we will still use a positive sign for the incomes (or receipts, or cash inflows) and a negative sign for the outcomes (or outlays, or cash outflows) from the point of view of the subject who evaluates. We should also recall that an operation at a given rate is fair if its balance at a given time (and then at all times if the law is decomposable) is zero. However, we have seen in section 4.1 that the fairness of an operation depends not only on its supplies, but also on the used exchange law z. If we adopt an exponential exchange law, this is identified by the annual rate i.

The rate i^* of the CCI (or exponential) exchange law that makes the given operation O fair, i.e. which makes (4.4) zero, is called the *internal rate of return* (IRR)[5]. It summarizes the return of the project made in O and, as we will see, is normally used as the basis of a decisional criterion on financial projects.

Let us analyze the concept of IRR to better clarify its meaning and its limits as a return measure. We have already shown that in a simple financial operation, of pure exchange, made of the supplies $(T_1,-C)$, $(T_2,+M)$, the percentage variation $(M-C)/C$, i.e. the related rate per period, is a measure of the return of the operation. More generally, for a complex operation O, defined in (4.1) or (4.1') with $n>2$, the IRR of the operation $O = \{T_h\}\&\{S_h\}$ is generalized as mentioned, as the interest rate of the exponential operation that makes O fair.

---

5 We note that with respect to O an operator can be both the investor and the borrower. We also observe that $V(T;O,z) = 0$ with an exponential law at rate i^* implies, due to the decomposability of the law, a compensation between payments and incomes of O taking into account the interest at the rate i^* and the current balances.

The meaning given to the internal rate, as a parameter whose use assures the fairness to the operation, implies that:

– such a parameter can be used to measure the investment return (or the financing cost) in the sense that summarizes the instantaneous returns, also a variable in time, in their evolution in the time interval of *O*;

– therefore, it corresponds to a constant instantaneous intensity in such a time interval, which becomes an exponential financial law.

Indeed we have seen in section 4.1 that if *O* is fair in cci at the rate i^*, clearly *the discount factor i*=IRR is also the return rate* inherent in the supplies of *O* together with the interest on the current reserves. We observe that it is not necessary that the retro-reserves (coinciding with the pro-reserves if *O* is fair) keep their sign in the time (i.e. that *O* is a *pure project*). The property also holds in the case of sign alternation (*O* = *mixed project*), as long as i^* is a reciprocal rate, i.e. it is valid both for earned and passive interest[6]. Using (4.4) with $i=i^*$ we obtain

$$- S_1 = \sum_{h=2}^{n} S_h \, (1+i^*)^{-(T_1 - T_h)} \tag{4.4''}$$

i.e. *O* can be interpreted, considering the case $S_1>0$, as the investment of S_1 at time T_1 that gives rights to the supplies (T_h,S_h), $h=2,...,n$, and i^* is the return rate of the operation.

If the payments are *periodic*, it is not restrictive to assume a *unitary* period (changing the unit measure and using the equivalent rate in the compound regime). In such a case, using $v = (1+i)^{-1}$, owing to (4.4) the calculation of IRR starts from the equation

$$V(0;O,i) = \sum_{h=1}^{n} S_h \, v^h = 0 \tag{4.19}$$

i.e. an algebraic equation of degree n in the unknown v. From the solution v^* the IRR $i^* = 1/v^* - 1$ is obtained. Well known theorems give information on the solutions of (4.19) in relation to the coefficients S_h.

6 For the general *mixed project* ruled by a non-reciprocal rate, it is necessary to use more general methods, which will be discussed in section 4.4.6. The interpretation of IRR as a return index in the mixed projects also implies a more rigorous widening of its meaning. This would lead to giving a return meaning to the IRR only when it is an index of totally detached or totally incorporated return, and only in the case of uniqueness of IRR for a given operation (see footnote 7).

The existence and uniqueness of the IRR for the operation O is not always verified. However, in order that the problem of IRR is mathematically *well posed* and financially meaningful as a return index, it is necessary that the solution v^* (and then i^*) exists and is unique. In this case we will say that i^* is IRR *operative*. This eventuality is verified in some special type of investments (or also of financings, obtainable reversing the algebraic sign of investment amounts), in which there is only one sign inversion in the sequence of amounts S_h. We will consider this later.

The value $V(0;O,i)$ is called the *discounted cash-flow* (DCF) and i^* is the rate that makes the DCF become zero.

4.4.2. *Outline on GDCF and "internal financial law"*

The concepts of DCF and IRR can be generalized going from a flat structure of interest rates to any structure, as long as it follows an s.dec financial law.

If we consider a financial operation O made of the amounts $s_0, s_1,...,s_n$ paid at increasing time $t_0, t_1,...,t_n$ (= intervals from a given time origin) then $O = \bigcup_{h=1}^{n}(t_h, s_h)$. We know that according to the signs of s_h, O can be an investment or a financing. Let us evaluate according to a exchange law $z(\xi,\eta)$ that, using $x<y$, becomes $z(y,x) = a(y,x)$ for discount from y to x, while $z(x,y) = m(x,y)$ for accumulation from x to y, being $m(x,y) = 1/a(y,x)$ in the symmetric hypothesis. Let us only consider discounting. Then the functional $G(\mathbf{a}) = \sum_{h=0}^{n} s_h\, a(t_h, t_0)$ depends on the function a and is called the *generalized discounted cash-flow* (GDCF) of O in t_0.

If $G(\hat{\mathbf{a}}) = \sum_{h=0}^{n} s_h\, \hat{a}(t_h, t_0) = 0$ results, the discount law is called the *internal financial law* (IFL) for O and is identified with $\hat{a}(t_h, t_0)$, defined in the payment times. Using IFL we obtain the fairness of O, i.e. the balancing between income and payments valued in t_0.

If $\hat{a}(t_h, t_0) = (1+i^*)^{-(t_h - t_0)}$, the GDCF gives rise to DCF and the IFL gives rise to the only parameter i^* (=IRR), that is the *internal rate of return* for O.

If the law $a(y,x)$ is s.dec and symmetric, we have $a(y,x) = 1/m(x,y)$ and the fairness does not depend on the evaluation time. Given the payment times $\{t_0, t_1,...,t_n\}$, we put

$$a_h = a(t_{h-1}, t_h)\ ;\ m_h = m(t_{h-1}, t_h),\ h=1,....,n \tag{4.20}$$

where $m_h = 1/a_h$.

Let us consider again the retro-reserve for *O*, which coincides with the pro-reserve because of the fairness. Let us indicate by c_h the retro-reserve calculated at time t_h *just after* the transaction s_h, which represents the credit obtained in t_h due to the previous transactions. The extreme values of the sequence $\{c_h\}$ are constrained by:

– $c_0 = -s_0$, having no earned interest yet;

– $c_n = 0$, due to the fairness of *O* valued with the IFL.

For the other values there exists a wide flexibility, connected to the choice of the accumulation factors $\{m_h\}$.

The following fundamental theorem holds.

THEOREM.– *For all operations O consisting of the cash-flow* $\bigcup_{h=1}^{n}(t_h,s_h)$, *each sequence* $\{c_h\}$ *of retro-reserves, with* $c_0 = -s_0$, $c_n = 0$ *under the condition* $c_h \neq 0, \forall h<n$, *gives rise biuniquely to a sequence of per period accumulated factors* $\{\hat{m}_h\} = \{\hat{m}(t_{h-1},t_h)\}$ *that form an IFL for O, owing to the recurrent system*

$$c_h = c_{h-1}\, m_h - s_h\;,\; h = 1,....,n. \tag{4.21}$$

Proof

The evidence of the biunique correspondence is based on the fact that the constraints give rise to a system of determined equations, which, given the supplies (t_h,s_h), identifies $\{m_h\}$ as a function of $\{c_h\}$ and vice versa. We can prove that $\{m_h\}$ is an IFL for *O* proceeding by induction on *h*. In fact

$$c_h = c_0\prod_{z=1}^{h} m_z - \sum_{u=1}^{h} s_u \prod_{z=u+1}^{h} m_z = c_0\, m(t_0,t_h) - \sum_{u=1}^{h} s_u\, m(t_u,t_h) \tag{4.22}$$

due to the decomposability of the financial law. Using $h=n$ and taking into account the constraints on the extreme of $\{c_h\}$, we obtain the fairness condition

$$\sum_{u=0}^{n} s_u\, m(t_u,t_n) = 0 \tag{4.23}$$

satisfied by the sequence $\{\hat{m}(t_{h-1},t_h)\}$, which is IFL for *O*. Going backwards, we can prove the opposite. From the factors $\hat{m}_h$ forming an IFL, we can obtain the intensities for the interval (t_{h-1}, t_h)

$$j_h = (\hat{m}_h)^{1/(t_h - t_{h-1})} - 1 \tag{4.24}$$

that give rise to the corresponding annual return.

4.4.3. *Classifications and property of financial projects*

If the retro-reserve of a project at the initial time 0 is zero, the value V at this time obviously coincides with the pro-reserve W.

If $V(i)$ is a decreasing function of i in the interval $(0,+\infty)$ and if (from the point of view of the investor)

$$\begin{cases} \lim\limits_{i \to 0^+} V(i) = \sum_h S_h > 0 \\ \lim\limits_{i \to +\infty} V(i) = S_0 < 0 \end{cases} \tag{4.25}$$

this is a sufficient condition for existence and uniqueness of a positive solution $i=i^*$ of (4.19) that gives the internal rate (sometimes called *implicit rate*), while its definition has no operative meaning without uniqueness.

The previous sufficient condition implies that for the investor the initial supply is a payment and that the algebraic sum of the amounts of O is positive, i.e. there exists a reward given by the surplus between incomes and payments. Starting the project with a payment (or an income) is in fact a characteristic of *investment operations* (or *financing operations*). The following definitions hold.

A project is called:

– *investment in the strict sense*, if all payments come before all incomes;

– *investment in the broad sense*, if the average maturity (defined in section 2.5.2) of payments comes before that of incomes at any evaluation rate.

We have a *financing (in strict or broad sense)* if the aforementioned definitions hold after the inversion of the sign of the amounts.

A project can be characterized by input and output amounts paid only one time or spread over more times, giving rise to four possibilities:

1) PIPO (= point, input, point output);

2) CIPO (= continuous input, point output);

3) PICO (= point input, continuous output);

4) CICO (= continuous input, continuous output).

For PIPO and PICO an investment project is *simple* if it is formed by one payment followed only by incomes; symmetrically a financing project is *simple* if it is formed by one income followed only by payments. For the simple projects of investment (or financing), the decreasing monotonic (or, respectively, increasing monotonic) *V(i)* is assured, and if

$$\left[\sum_h S_h + S_0 > 0\right] \cap \left[S_0 < 0\right] \text{ or } \left[\sum_h S_h + S_0 < 0\right] \cap \left[S_0 > 0\right]$$

the operative internal rate exists.

Considering investment projects, it is interesting to give more general conditions for the existence for the operative IRR. It can be proved that in the hypothesis of $V(0) > 0$ of the project, besides the simple investment PIPO and PICO projects (which are investment in strict sense), there is existence and uniqueness of the solutions of (4.19) also in the following cases:

– in the other investment projects in the strict CIPO and CICO sense;

– in the investment project in the broad sense, of type CICO, when the condition (also sufficient because the project is an investment in the broad sense) that the arithmetic mean of the time of payments (= their average maturity when $i \to 0$) comes before the time for the first income (= average maturity of incomes when $i \to +\infty$) is satisfied.

Indeed, in both cases *V(i)* is decreasing until it remains positive, and approaches $S_0 < 0$ when *i* diverges, which gives the existence and uniqueness of its roots.

The IRR has the following properties:

– it does not change with a proportional change of the amounts;

– the project sum of two projects with internal rate has an internal rate with value between the rates of the two projects, then, if the two rates coincide, the rate of the project sum coincides with them.

If the aforementioned conditions for the average maturities are not satisfied, we can lose the existence or uniqueness of the roots of (4.19), no longer having the possibility of defining an operative IRR. Reversing the sign of the amounts in the cash-flow, the investment becomes a financing. Then the decreasing of *V(i)* changes to increasing and the return rate is a cost rate for the financing, maintaining the previous property.

Summarizing the operative interpretation of the IRR of an investment project, this rate is (if it exists) a return index, because it is just the interest rate of a profitable account fed only from the financial transactions connected with the project, such that, also considering the interest, the balance is zero just after the last transaction[7].

EXAMPLE 4.2.– We give here two examples, regarding cash-flow of investment projects, for which the properties defined above are satisfied. To consider financing projects of the same type, it is enough to reverse the sign of the monetary amounts:

– a project of *investment in the strict sense* of type CICO, called $\mathcal{A}$, is as follows. In an industrial plant the following costs and revenues apply: for the first 2 semesters only costs apply; so €70,000 in the 1st semester for buying the plant, and €40,000 in the 2nd semester for installations in the plant; in each of the following 10 semesters we have operating costs for €6,000 and income for €19,000; in the 13th semester the divestment of the plant occurs with a net return of €15,000. The algebraic sum of these transactions is €35,000, thus it is a profitable investment. Valuing the amounts at the end of each semester, the cash-flow of $\mathcal{A}$ is shown by the following graph:

-70,000	-40,000	13,000	13,000	....	13,000	13,000	15,000
1	2	3	4		11	12	13

$\mathcal{A}$ is an investment project in the strict sense, with a unique IRR, because when balancing in the year incomes and payments, the time of the last (net) payment comes before that of the first (net);

– an *investment project in the broad sense* assumes an average maturity of payments preceding that of incomes (in net terms) at whatever evaluation rate is used; for this it is sufficient condition that the arithmetic mean maturity of payments (= their average maturity when $i \to 0$) comes before the time of first income (= average maturity of incomes when $i \to +\infty$).

7 Here it is assumed that, if the balance does not remain constant in sign due to the dynamic of the financial supplies, the allowed interest rate is the same as the charged rate, i.e. the c/a bears reciprocal rate. In some cases, this is a strong limitation, the overcoming of which requires a more general approach (see section 4.4.6).

An example of such a type of project can be obtained by modifying the project $\mathcal{A}$ in $\mathcal{B}$ adding a payment of €15,000 after 6 semesters. Compensating with a net operating income in the period for €13,000 we obtain for the 6th semester a net payment of €2,000 and then the cash-flow of $\mathcal{B}$ can be described by the following graph:

```
-70,000 -40,000 13,000 13,000 13,000 - 2,000 3,000          13,000  15,000
_|______|______|______|______|______|______|___............_|______|___
 1      2      3      4      5      6      7                12      13
```

Measuring in semesters, the arithmetic mean maturity of $\mathcal{B}$ payments is:

$$(-70{,}000 - 40{,}000 \cdot 2 - 2{,}000 \cdot 6)/(-112{,}000) = 1.4469$$

whereas the time for the first income is 3 and then the project $\mathcal{B}$ is an investment in the broad sense with unique IRR; it is profitable because the algebraic sum of the amounts is +5,000.

4.4.4. *Decisional criteria for financial projects*

It is fundamental in mathematical finance to give a criterion to decide if it is convenient or not for an economic subject to realize a project identified by the financial operation *O*[8].

Not considering a particular criterion based on parameters and particular points of view, we will focus our attention on the two more important criteria as they are better justified in general in the light of financial equivalencies and are universally used in business practice.

The first criterion is based on the *value V* of *O* in a given time of evaluation[9]. This has been defined in this chapter and its meaning is clear in quantifying *O*. The second criterion is based on the *internal rate of return* of *O*, already defined, if it is operative.

Usually the first criterion is considered *subjective*, because *V* depends on the evaluation rate *x* that is subjectively fixed by the decision maker, while the second criterion is considered *objective*, because the internal rate depends only on objective elements, such as the fixed supplies of *O*. However, looking carefully at both

8 For further discussion on the internal rate and, in general, on decision and choice in financial projects, see in Italian literature: Levi (1967); Trovato (1972).
9 If the valuation time is not after the beginning of the operation or if the supplies before such time do not matter, the value *V* can be changed with the prospective reserve of *O*.

criteria, as we will see later, the decision follows from a comparison between an *external* value (due to the *market*, which implies a *subjective* evaluation by the decision maker) and an *internal* value (connected with the *objective* features of the *project*, i.e. its supplies or internal rate).

It is clear that the evaluation depends on the choice of the financial law, but if, as is usually the case, an exponential regime is used, a feature of which is a flat structure of rates, it is sufficient to fix the annual rate to identify the exponential law. In this way the decision is not influenced by a changing of the evaluation time and not even by a uniform deferment of the financial supplies.

We can therefore enunciate the *first decisional criterion* in the following way.

CRITERION OF THE PRESENT VALUE (PV): *the project identified by a financial operation O* of investment or financing is convenient for the economic subject "A", and then the decision on it is positive, if and only if, at the evaluation rate x* chosen by "A", the value V of O* at the evaluation time results in:* $V(x^*) > 0$. *The project identified by O* is not convenient, and then the decision on it is negative, if and only if in the same conditions the result is:* $V(x^*) < 0$. *Finally, if and only if* $V = 0$ *the project is indifferent*[10].

This is reasonable according to the profit criterion: if and only if the cash-flow of the projects implies withdrawals and deposits on a profitable account at the (reciprocal) rate x^* (that – taking into account the received and allowed interest on the balance, initially zero, that is accruing from time to time – give rise to a final positive spread, for which the discounted evaluation $V(x^*)$ remains positive), the project is convenient. Otherwise it is unacceptable or, at most, indifferent.

From the above the criterion to choose the evaluation rate immediately follows: it is necessary to choose the *market rate* of the financial operations, which are alternatives to the examined project.

We could not add anything on the projects decision, given the overall validity of the PV criterion and its dependence on the fundamental principles of financial

10 The decisional criterion can be extended to many projects, if the decision maker has available funds and he is interested in sustaining more than one, in the following way: *given* $n \geq 2$ *investment projects* $O_1,\ldots,O_n$, *each being convenient according to the evaluation rate* x^* *(concerning the opportunities of financing the projects, or by vanishing profit or by rising cost), let us put them in decreasing order of their value, such that* $V_1(x^*) > \ldots > V_n(x^*)$. *For the decision maker it is convenient to carry on the first* $r \leq n$ *projects with values* $V_1(x^*),\ldots, V_r(x^*)$ *for which he has enough funds. He can also add one of the subsequent projects, in convenience order, if it can be split (as, for example, is done with stocks or company's share), thus buying a part of the project.*

equivalency and on the arbitrariness of the choice of the evaluation law. Furthermore, we need to take into account the common wish of economic operators, especially in the business world, to fix a criterion on an objective basis. This explains the wide spread of the *second decisional criterion*, which can be given as follows.

CRITERION OF THE INTERNAL RATE OF RETURN (IRR): *if an investment project, identified by an operation O^*, with internal rate i^* (to be considered like return rate), is convenient, then the decision is positive, if and only if $i^* > x^*$, where x^* is the evaluation rate, in particular the market rate (to be considered like external rate of the financing costs needed for the investment). If O^* is not convenient, then the decision is negative, if and only if $i^* < x^*$.*

For the financing projects it is enough to change signs and the inequalities side, and the following formulation holds: if a financing project O^*, with internal rate i^* (to be considered likecost rate) is convenient, then the decision is positive, if and only if $i^* < x^*$, where x^* is the evaluation rate, in particular the market rate (to be considered like external rate of return of the investments following the financing). If O^* is not convenient, then the decision is negative, if and only if $i^* > x^*$.

Regarding the criterion IRR to decide on a single project, we observe that:

– it does not have an overall validity, because it assumes the existence of an operative internal rate for O^*;

– the arbitrariness is not eliminated because in any case it is necessary to choose the external rate x^*, in order to compare it with i^*;

– the foundation of the criterion, when it is enforceable, comes from that of the present value criterion, as follows by the proof here schematized for investment or financing projects with a positive internal rate.

If $V(i)$, which is strictly decreasing or increasing depending on the project being an investment or a financing, has only one root i^*, the following inequality couples are equivalent (in the sense that one is necessary and a sufficient condition for the other):

– *investment projects:*

$V(x^*) > 0 \Leftrightarrow i^* > x^*$: O^* = convenient investment

$V(x^*) < 0 \Leftrightarrow i^* < x^*$: O^* = non-convenient investment

– *financing projects:*

$V(x^*) > 0 \Leftrightarrow i^* < x^*$: O^* = convenient financing

$V(x^*) < 0 \Leftrightarrow i^* > x^*$: O^* = non-convenient financing

Comment

The positive decision on the project identified by O^* is equivalent to the choice of O^* instead of the "no project" (that is, the project of doing nothing), featured by the absence of cash-flow and then by the maintenance of the "status quo ante", according to which the wealth of the economic subject was profitably invested, for example at the evaluation rate x^*. The criterion IRR in the previous formulation then follows, because "doing O^*" means the "transferring financial funds", from the market to the project if O^* is an investment or from the project to the market if O^* is a financing.

EXAMPLE 4.3.– *Evaluation of investment projects. Calculation of IRR. Decisions.*

We will consider three investment projects, all having an operative IRR.

A) The simple investment project $\mathcal{A}$ of the PICO type is the buying with cash at time 0 of a real estate equipped with an industrial plant that is producing a detached return, with a sale after 5 year that implies an incorporated return. Let the purchase price be € 47,500, the semiannual return balanced at the end of term, after tax and operating expenses, is €3,000, the selling price at the end of the 5^{th} year is €50,000. It is then a financial project featured by the following supplies:

-47,500	+3,000	+3,000	.........+3,000	+53,000	(amounts)
0	1/2	1	9/2	5	(time)

Using "semester" as a measure of time and indicating with x the semiannual evaluation rate, the PV (or DCF) of the project $\mathcal{A}$, given by the initial value of its supplies, is expressed by

$$V(x) = -47{,}500 + 3{,}000\,[(1+x)^{-1}+\ldots+(1+x)^{-10}] + 50{,}000\,(1+x)^{-10}$$

which results in:

$$V(0.062) = 1{,}770.626;\ V(0.07) = -\,1{,}011.791$$

Decisions according to PV criterion. By using money borrowed at a semiannual rate of 6.2% (for self-financing by *vanishing profit* or external financing by *rising cost*) the investment is convenient; however, by using money borrowed at a semiannual rate of 7% the investment is not convenient.

Calculation of semiannual IRR. The semiannual IRR, unique and certainly between 0.062 and 0.07, will be calculated first of all, obtaining an estimation through linear interpolation in the interval (0.062, 0.07) and then, to obtain the solution (with 9 exact decimals required), proceeding with a classical iteration (for more detail on these methods, see section 4.5) starting from the approximate solution, using Excel spreadsheets[11].

The linear interpolation leads to solving the equation

$$\frac{x - 0.062}{0.07 - 0.062} = \frac{0 - 1{,}770.92}{-1{,}011.79 - 1{,}770.92}$$

with solution $\bar{x}$ = 0.067091. To obtain the exact solution $\hat{x}$ with 9 decimals, we apply to the classical iteration the transformation "$f(x) = xg(x)/g_0$" to go from equation "$f(x) = x$" to the equivalent equation "$g(x) = g_0$", which is more useful here, thus using

$g(x) = V(x) + 47{,}500$ (= present value of incomes); $g_0 = 47{,}500$ (= initial payment)

With an Excel spreadsheet we build the following five columns. The resulting Excel table is as follows.

11 We notice that Excel has included a function for the calculation of IRR, starting from a given cash-flow with periodic payments. Furthermore, for calculating the IRR by means of the Excel method we need a starting approximated evaluation, which can be obtained with the formula in footnote 22 of this chapter regarding the linear interpolation method. However, for the calculation of IRR, it is useful to give an illustration of classical numerical methods.

0	0.067091000	47,475.956	0.999494	0.067057039
1	0.067057039	47,487.647	0.999740	0.067039600
2	0.067039600	47,493.652	0.999866	0.067030641
3	0.067030641	47,496.737	0.999931	0.067026037
4	0.067026037	47,498.323	0.999965	0.067023670
5	0.067023670	47,499.138	0.999982	0.067022454
6	0.067022454	47,499.557	0.999991	0.067021829
7	0.067021829	47,499.772	0.999995	0.067021508
8	0.067021508	47,499.883	0.999998	0.067021342
9	0.067021342	47,499.940	0.999999	0.067021257
10	0.067021257	47,499.969	0.999999	0.067021214
11	0.067021214	47,499.984	1.000000	0.067021191
12	0.067021191	47,499.992	1.000000	0.067021180
13	0.067021180	47,499.996	1.000000	0.067021174
14	0.067021174	47,499.998	1.000000	0.067021171
15	0.067021171	47,499.999	1.000000	0.067021169
16	0.067021169	47,499.999	1.000000	0.067021168
17	0.067021168	47,500.000	1.000000	0.067021168
18	0.067021168	47,500.000	1.000000	0.067021168

Table 4.3. *Intermediate calculations for the case of A*

The first column is built with natural numbers (i = number of steps) and the 1st row (i=0) has components: 0; $x_0 = \overline{x} = 0.067091$; $g(x_0)$; $g(x_0)/g_0$; $f(x_0)$. The 2nd row (i=1) starts with: 1; $x_1 = f(x_0)$; the remaining part is built, using the "copy and paste" function, by columns. In the 3rd, 4th and 5th columns the values $g(x_i)$; $g(x_i)/g_0$; $f(x_i)$ for $i \geq 1$ are obtained; and then in the 2nd column we obtain the values x_i for $i \geq 2$, that are the sequence converging to the solution $\hat{x} = 0.067021168$. We get such value by the 17th iteration, because $g(x_{17}) = 47{,}500$ and at the 18th iteration we can see that with 9 decimals $x_{18} = x_{17}$ and then such a value is $\hat{x}$. The compound annual IRR is $(1+\hat{x})^2 - 1 = 0.138534$.

The Excel instructions are as follows: A1: 0; B1: 0.067091; C1: = 3000*(1-(1+B1)^-10)/B1+50,000*(1+B1)^-10; D1: = C1/47500; E1: = B1*D1.

A2: = A1+1; B2: = E1; copy B1,C1,D1,E1,A2, then paste on the subsequent elements of the same column.

Decisions according to IRR criterion. If the annual evaluation rate (in practice the market rate by *vanishing profit* or by *rising cost*) is less than 13.8534%, the decision on the project is positive; otherwise it is negative.

B) The pure investment project $\mathcal{B}$ in the strict sense of CICO type, and then with unique IRR, is as follows. Let us consider an industrial plant which gives: for the first 2 years only payments, so a purchase cost of €7,000 in the 1st year and an installation cost of €4,000 in the 2nd year; for the next 5 years in operative phase one has annual managing costs of €1,200 and annual incomes of €3,800; for the next year the plant is sold for a net return of €1,500. The algebraic sum of the transactions is €3,500 and then we have a profitable investment. Valuing the amounts at the end of each year, the cash-flow of $\mathcal{B}$ is expressed by

-7,000	-4,000	+2,600	+2,600	+2,600	+2,600	+260	+1,500	(amounts)
1	2	3	4	5	6	7	8	(time)

Using $v = (1+x)^{-1}$, the PV is expressed by

$$V(x) = -7{,}000\, v + [-4{,}000 + 2{,}600(v+v^2+v^3+v^4+v^5)]v^2 + 1{,}500\, v^8$$

and then

$$V(0.07) = +148.51 \quad ; \quad V(0.08) = -200.36$$

Decisions according to PV criterion. If money is borrowed at the annual rate of 7%, the investment is convenient, but if money is borrowed at the annual rate of 8% (both rates by *vanishing profit* or by *rising cost*) the investment is inconvenient.

Calculation of IRR. The IRR, which is unique and certainly between 0.07 and 0.08, will be calculated starting with an approximation through linear interpolation in the interval (0.07; 0.08) and then, to obtain the solution (with 9 decimals required), with a classical iteration starting from the approximate solution. The linear interpolation leads to the solution of the following equation:

$$\frac{x-0.07}{0.08-0.07} = \frac{0-148.51}{-200.36-148.51}$$

This (approximate by excess) solution is $\bar{x} = 0.074257$. In order to obtain the exact solution $\hat{x}$, we apply the classical iteration method. Furthermore, we cannot start from the equation $x = f(x)$, where $f(x) = x + V(x)$, because in a neighborhood of $\hat{x}$ is $|f'(x)| > 1$. We need to start from the equation $x = h(x)$ (equivalent to $x = f(x)$) where $h(x) = [f(x)-mx]/[1-m]$. We should use $m = f'(\hat{x})$ to obtain $h'(\hat{x})=0$ and an immediate convergence, but this is not possible because $\hat{x}$ is unknown. Furthermore, in order to obtain the convergence with the sequence $\{x_i\}$ obtained by $x_{i+1} = h(x_i)$, it is enough that $|h'(\hat{x})| < 1$, for which m must be a well approximated value of $f'(\hat{x})$. We can use: $m = \Delta f/\Delta x = 1+\Delta V/\Delta x$ in a neighborhood of $\hat{x}$. Thus, we obtain the following function, suitable for the iteration

$$h(x) = \frac{V(x) - x - \left[1 + \frac{\Delta V}{\Delta x}\right]x}{1 - 1 - \frac{\Delta V}{\Delta x}} = \frac{V(x) - x\frac{\Delta V}{\Delta x}}{-\frac{\Delta V}{\Delta x}}$$

We assume $\Delta V/\Delta x$ on the interval (0.07; 0.08) containing $\hat{x}$, thus using some previous calculations: $\Delta V/\Delta x$ = [-200.36 -148.51]/ [0.08-0.07] = -34887 results, which is to be substituted in the previous expression for $h(x)$. Given that, let us use Excel and proceed as in section A of this example. We obtain the results in Table 4.4.

0	0.074257000	-3.22391125	0.074164590
1	0.074164590	0.01840163	0.074165117
2	0.074165117	-0.00011144	0.074165114
3	0.074165114	0.00000067	0.074165114
4	0.074165114	0.00000000	0.074165114

Table 4.4. *Intermediate calculations for the case of B*

The 1st row is the vector with components: 0, $\bar{x}$, $V(\bar{x})$, $[V(\bar{x})+34{,}887\,\bar{x}]/34{,}887$; the 2nd row starts with the values 1, $[V(\bar{x})+34{,}887\,\bar{x}]/34{,}887$ and the remaining part of the table is completed using the copy and paste function. The 2nd column shows the rate sequence converging to the solution; already in the 4th iteration we obtain $\hat{x}$= 0.074165114 with 9 exact decimals and $V(\hat{x}) = 0$.

The Excel instructions are as follows:

– A1: 0; B1: 0.074257;

– C1: = -7,000*(1+B1)^-1+(1+B1)^-2*(-4,000+2,600*(1-(1+B1)^-5)/B1+1,500*(1+B1)^-8;

– D1:= (C1+3,4887*B1)/34,887;

– A2: = A1+1; B2: = D1; copy B1, D1, A2, then paste on the following column cells.

Decisions according to IRR criterion. If the annual evaluation rate is less than 7.4165114%, the decision on the project is positive; otherwise it is negative.

C) Let us consider the project obtained modifying $\mathcal{B}$ with the addition of repairing costs after 6 months for €3,000. Compensating with the net operating income for €2,600, at the 6th year we have a net payment of €400 and the cash-flow is given by

-7,000	-4,000	+2,600	+2,600	+2,600	-400	+2,600	+1,500	*(amount)*
1	2	3	4	5	6	7	8	*(time)*

The new project is not an investment in the strict sense, but it verifies the sufficient condition in order that it is an investment in the broad sense and the IRR exists and is unique. In fact, the arithmetic averaged maturity of payments is (-7,000 -4,000·2 - 400·6)/(-11,400) = 1.53, and the time of first income is 3. Furthermore, the project is profitable because the algebraic sum of the monetary transactions is +500. Using $v = (1+x)^{-1}$, the PV is given by

$$V(x) = -7{,}000\, v + (-4{,}000 + 2{,}600\, (v+v^2+v^3+v^5)]\, v^2 - 400\, v^6 + 1{,}500\, v^8$$

and

$$V(0.01) = +77.49\ ;\ \ V(0.07) = -311.98.$$

Decisions according to the PV criterion. If money is borrowed at the annual rate of 1% (by *vanishing profit* or by *rising cost*) the investment is convenient, but if money is borrowed at the annual rate of 2% the investment is inconvenient.

Calculation of the IRR. We proceed in the same way as for $\mathcal{B}$, having the same conditions. The IRR, which is unique and between 0.01 and 0.02, will be calculated starting from an approximated value estimated using linear interpolation in the interval (0.01; 0.02) and then, to obtain the solution (with 9 exact decimals required), proceeding with classical iteration starting from the approximated solution. The linear interpolation leads to the following solution of the equation:

$$\frac{x-0.01}{0.02-0.01} = \frac{0-77.49}{-311.98-77.49}$$

and then the solution $\bar{x}$ = 0.011989. To obtain the exact solution $\hat{x}$, we apply the classical iteration method, with the same transformation and procedures applied in section B of this example. The function to use for the iteration on the equation $x = h(x)$ is

$$h(x) = \frac{V(x) - x\frac{\Delta V}{\Delta x}}{-\frac{\Delta V}{\Delta x}}$$

We assume $\Delta V/\Delta x$ in the interval (0.01;0.02) containing $\hat{x}$: we obtain

$$\Delta V/\Delta x = (-311.98 - 77.49)/(0.02 - 0.01) = -38947$$

and such a value is to be substituted in the previous expression for *h(x)*. As a result, using Excel as in section A, the following table is obtained.

0	0.011989000	1,426.860635	-1,429.377990	-2.517356	0.011924365
1	0.011924365	1,429.813919	-1,429.751608	0.062310	0.011925964
2	0.011925964	1,429.740801	-1,429.742360	-0.001559	0.011925924
3	0.011925924	1,429.742630	-1,429.742591	0.000039	0.011925925
4	0.011925925	1,429.742584	-1,429.742585	-0.000001	0.011925925
5	0.011925925	1,429.742585	-1,429.742585	0.000000	0.011925925

Table 4.5. *Intermediate calculations for the case C*

The 1st row is the 6 component vectors: 0, $\bar{x}$, two addends of $V(\bar{x})$ needed for the calculation, $V(\bar{x})$, $[V(\bar{x})+38{,}947\,\bar{x}]/38{,}947$; the 2nd row starts with the values 1, $[V(\bar{x})+38{,}947\,\bar{x}]/38{,}947$ and the remainder of the table is completed with the copy and paste function. The 2nd column shows the sequence of rates converging till the solution; at the 5th iteration we obtain $\hat{x}$= 0.011925925 with 9 exact decimals and $V(\hat{x}) = 0$.

The Excel instructions are as follows.

– A1: 0;

– B1: 0.011989;

– C1: = -7,000*(1+B1)^-1+(1+B1)^-2*(-4,000+2,600*(1-(1+B1)^-5)/B1);

– D1: = -3,000*(1+B1)^-6+1,500*(1+B1)^-8;

– E1: = C1*D1;

– A2: = A1+1;

– B2: = F1; copy C1,D1,E1,A2,B2, then paste on the subsequent column cells.

Decisions according to the IRR criterion. If the annual evaluation rate (really the market rate by *vanishing profit* or by *rising cost*) is less than 1.1925925%, the decision on the project is positive; otherwise it is negative.

Comparison between Example 4.3B and Example 4.3C. The input in $\mathcal{B}$ of the payments of €3,000 at the 6th year reduces the convenience threshold from 7.42% to 1.19% in terms of the highest acceptable rate, making the investment C almost inconvenient at the current rates.

4.4.5. *Choice criteria for mutually exclusive financial projects*

In section 4.4.3 we considered a project as the only alternative to *no project*. However, it can be necessary to make a *choice between two projects*, which are each convenient on their own, but not together[12].

For this particular problem we can use a criterion based on the present value or otherwise on the internal rate. However, we first need to observe that a coherent choice implies *comparability* between the cash-flows of the mutually exclusive project in homogenous conditions; we use in this case the expression *complete alternative*, which, in the case of PIPO or PICO investment projects with only one initial payment, means:

– same initial payment;

– same time length.

---

12 We stress that the external comparison rate to judge the convenience of two projects could not be the same, with a choice based on the circumstances. Then if the first project can be realized with self-financing, disinvesting a previous investment, the internal rate must be compared with its return rate (by vanishing profit), whereas if the second project can be realized only by borrowing money, its convenience must be evaluated by comparing the internal rate with the external rate of the financing (by rising cost), which is usually greater than the return rate.

If there is not a *complete alternative* between the two projects, we need to consider *additional operations* of shorter initial cost or length for the projects, which allow a comparison between homogenous elements. In short – considering two projects – if O_1 and O_2 do not give rise to *complete alternatives*, in order to make the comparison it is necessary to go back to *complete alternatives*, considering two additional projects Q_1 and Q_2 so that the unions $O_1 \cup Q_1$ and $O_2 \cup Q_2$ (where $\cup$ is a union between projects) are a *complete alternative*[13].

Therefore, in order to choose between two projects, we can apply the following criterion.

PRESENT VALUE CRITERION. *Given two investment projects characterized by cash-flows with values $V_1(x^*)$, $V_2(x^*)$ positive according to an evaluation rate x^* (concerning the projects' financing opportunities, by vanishing profit or by rising cost), the decision maker who, in a complete alternative condition, can carry on only one of them chooses the project that gives rise to a higher value*[14]. *The same decisional criterion holds for mutually exclusive financing projects.*

The consideration of criteria based on the internal rate points out delicate questions that need to be clarified. Indeed, for the financial valuations only in the initial part of the company's life we can neglect the pre-existing conditions; when considering the alternative projects, we need to think in terms of substitutive projects and then consider the difference of cash-flows.

Indeed, if $\mathcal{P}$ represents the projects already realized by the considered company (assuming a past activity) and if the owner has to decide between two new projects $\mathcal{A}$ and $\mathcal{B}$ (both acceptable, if individually considered), the owner does not have to compare $\mathcal{A}$ and $\mathcal{B}$ but $\mathcal{P} \cup \mathcal{A}$ and $\mathcal{P} \cup \mathcal{B}$ and then the owner must choose, assuming a favorable preliminary decision on $\mathcal{B}$, whether or not to substitute $\mathcal{B}$ for $\mathcal{A}$ and then must decide on $\mathcal{A} - \mathcal{B} = (\mathcal{P} \cup \mathcal{A}) - (\mathcal{P} \cup \mathcal{B})$. In such a case the owner, after having decided to add $\mathcal{B}$ and $\mathcal{P}$, considers it more advantageous to withdraw such a decision (i.e. subtract $\mathcal{B}$ to go back from $\mathcal{P} \cup \mathcal{B}$ to $\mathcal{P}$, not considering costs) and then to add $\mathcal{A}$ to $\mathcal{P}$.

In summary, a choice in alternative between $\mathcal{A}$ and $\mathcal{B}$, both investment or both financing projects, is equivalent to a decision on $\mathcal{B} - \mathcal{A}$. Thus, the general extension of the IRR criterion to alternative choices with reference to the rates of the projects to be compared is not justified. Because of the overall validity of the evaluation, it is

13 It is not necessary to take into account Q_1 (or Q_2) if its internal rate equals the evaluation rate, because then the value increment due to the integration is zero.

14 Notice that a such criterion can be obtained as a particular case of the criterion described in footnote 10 using $n = 2$, $r = 1$.

indeed necessary to value the *difference project* of two alternative projects (considering that $\mathcal{A} - \mathcal{B}$ and $\mathcal{B} - \mathcal{A}$ have the same internal rate, if it exists operative, as defined in section 4.4.1) and to apply to such *difference project* the decisional criterion IRR (see Figures 4.2a and 4.2c).

Furthermore, in the particular case of *dominance* between projects, we will say that the project ***b*** is *dominant* over the project ***a*** if

$$V(0;\boldsymbol{b},x^*) > V(0;\boldsymbol{a},x^*)\ ,\ \ \forall x^* \in X^* \tag{4.26}$$

results (where X^* is the set of variation of all possible evaluation rates), there is no IRR for the difference operations. In addition

– with reference to *investment,* the project with higher value for each rate also has higher IRR. In particular, if (4.26) holds, because of the decrease of *V*, the inequality $i^*(\boldsymbol{b}) > i^*(\boldsymbol{a})$ concerning the IRR of **a** and **b** (see Figure 4.2b);

– with reference to *financing*, the project with higher value for each rate also has lower IRR. In particular, if (4.26) holds, because of the increase of *V*, the inequality $i^*(\boldsymbol{b}) > i^*(\boldsymbol{a})$ concerning the IRR of **a** and **b** (see Figure 4.2d).

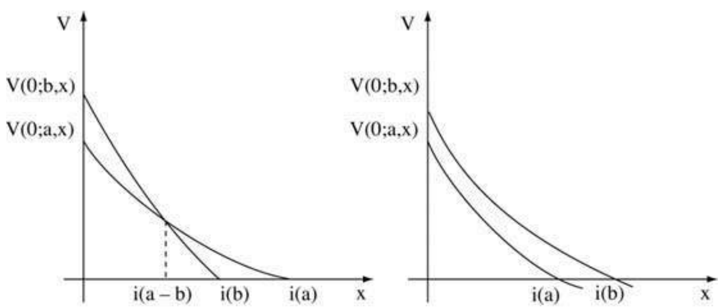

Figure 4.2a. *Investment projects* **Figure 4.2b**

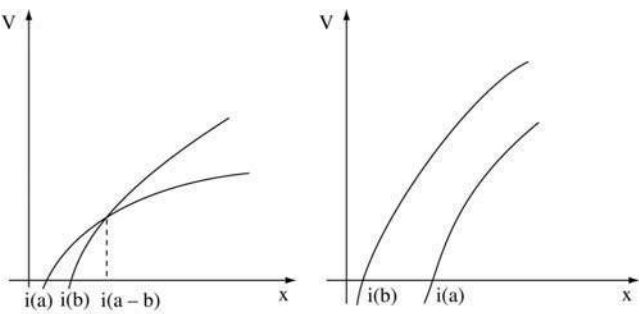

Figure 4.2c. *Financing projects* **Figure 4.2d**

In conclusion, for the alternative choices we can enunciate the following.

INTERNAL RATE CRITERION

1) Case of no dominance. Given two convenient investment projects **a** and **b** without dominance in the rates set X*, if there exists operative IRR i* of the difference operation **a-b** (and then of **b-a**) and if at the rate 0 is $V(0;\boldsymbol{b},0) > V(0;\boldsymbol{a},0)$, then, indicating by x* the external evaluation rate chosen by the decision maker, if x* < i*, **b** is preferred to **a**; if x* > i*, **a** is preferred to **b**.

The opposite inequalities hold if $V(0;\boldsymbol{b},0) < V(0;\boldsymbol{a},0)$.

2) Case of dominance. Given two convenient investment projects **a** and **b** in a dominance relation in the rates set X* and having operative IRR i*(**a**) and i*(**b**), the decision maker prefers the project with the higher IRR.

If the two projects to be compared are financing, their cash-flows are obtained from that of the investment projects changing the amounts' sign. So, the previous criterion holds except for an inversion of the inequalities between the rates considered in case 1), whereas the project with lower IRR is chosen in case 2).

4.4.6. ***Mixed projects: the TRM method***

In section 4.4.5 we clarified the relation connecting the choice between two alternative projects and the so-called *substitutive financial operations*, which are obtained formally as *difference operations* between two operations (of investment or financing). Such operations are used when, among other things, we want to cancel a project already chosen to substitute it with another project corresponding better to the new company's aim[15]. We saw that the choice between two projects that are both acceptable can lead back to a difference operation.

The next problem is moreover that a difference operation $\mathcal{A} - \mathcal{B}$ between two investment (or financing) projects having operative IRR is not always a project with operative IRR. This leads to careful discussions about the non-existence or plurality of solutions to the IRR problem for a substitutive project. However, it is an added basic issue which, when resolved, leads to canceling the discussed problems and some formulation defects. We wish to consider this problem.

The present value criterion described in sections 4.4.4 and 4.4.5 uses the evaluation rate referable to the received rates in the investment markets or the allowed rates in the financing markets. If the project is profitable for the

15 See Volpe di Prignano and Sica (1981).

entrepreneur, he uses his profits on the investment market at a received rate of return; however, if the project receives money from the entrepreneur, he gets it from the financing market at an allowed cost rate. Yet because the project to which the criterion is applied changes, as usual, between investment and financing periods, the received and allowed rates in the respective markets are usually different and it does not correspond to reality to use the same rate, i.e. operate at a *reciprocal rate*, as implicitly postulated by the criterion on decisions and choices problems. We then have to leave, for this type of project, the restrictive view followed so far and to introduce new definitions and formulations, also increasing the dimension of the variability space of the examined quantities.

With that aim we will now follow a formalized approach that has given rise to the TRM[16] method. The generalization in the approach will be clarified later.

In such an approach the entrepreneur is seen as an operator in intermediate position between *market* and *project*, which can be realized in an industrial (or commercial or financial or other) venture, which:

a) obtains money, as input to investments for the entrepreneur – who puts his own means, which were profitably invested, or is financed by the external money market, for the most the bank system – with a cost that in the first case is *lost profit*, and in the second case is *emerging cost*;

b) gives subsequently (but the cycle could be reversed, as in the insurance sector) output as money that the entrepreneur then invests in a profitable manner.

In the process described here, there are, in general, four different rates that are considered:

– a pair of rates, usually different[17], for the cost and profit of money, which are the received and allowed rates (for the entrepreneur) of *external* type on the money market, i.e. the cost rate r^* of financing or self-financing needed for investment in the project and the return rate k^* of the investments made with the profit from the project;

– a pair of *internal* rates, $\hat{r}$ to debit and $\hat{k}$ to credit of the project, i.e. that relate to the objective characteristics of the project. As we will see, this second pair is enlarged on an infinite set of equivalent pairs.

Let us assume a *discrete* approach, i.e. considering the supplies of the entrepreneur relative to the project and the following balances only periodically (in

16 See Teichroew, Robichek, Montalbano (1965a) and (1965b).
17 We will come back to this later.

particular, annually). In addition, let us indicate by $\{a_i\}$, (i = 0, 1,..., n) the cash-flow of the project (i.e. $a_i < 0$ for payments, $a_i > 0$ for income of the entrepreneur).

In the TRM method the following classification is introduced, relative to the dynamic of the cumulated current balance $S_t = S_t\ (k,r)$, (t= 0,1,..., n), of a c/a devoted to the examined project. From this c/a the inputs come out and to this c/a the outputs come in, both connected to the project[18].

A project is said to be *pure (at a given rate)* when accounting the financial transactions connected to a project in a profitable c/a at such a rate, the balances S_h, h=0,...,n-1 before the last transaction have the same sign, while $S_n = S_n(k,r)$ can be >=< 0, constituting the *final result* of the project (from the viewpoint of the entrepreneur).

Obviously:

– we have a *project of pure investment (at the investment rate r)* if the first financial transaction is a payment and, accounting for the interest charged to the project at the rate r, the balances S_h, h=0,...,n-1, remain ≤ 0. In formulae this is:

$$\begin{cases} S_t = S_{t-1}(1+r) + a_t\ , \quad t = 1,...,n \\ S_0 = a_0 < 0 \quad ; \quad S_h \leq 0\ \ , \ \ h = 1,...,n\text{-}1 \end{cases} \tag{4.27}$$

– we have a *project of pure financing (at the financing rate k)* if the first financial transaction is an income and, accounting for the interest charged to the project at the rate k, the balances S_h, h=0,...,n-1, remain ≥ 0[19]. In formulae this is:

$$\begin{cases} S_t = S_{t-1}(1+k) + a_t\ , \quad t = 1,...,n \\ S_0 = a_0 > 0 \quad ; \quad S_h \geq 0\ \ , \ \ h = 1,...,n\text{-}1 \end{cases} \tag{4.28}$$

A project is said to be *mixed (at the investment rate r and at the financing rate k)* if it is neither a pure investment at the rate r, nor a pure financing at the rate k. Thus, accounting the interest charged or favorable to the project at such rates, the balances S_h, h=0,...,n-1, do not remain of constant sign, and it can result in >=< 0, with alternating phases of investment and financing. In formulae

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18 Observe that the balance from the viewpoint of the entrepreneur is equal to the retro-reserve from the viewpoint of the project, identified as a counterpart of the entrepreneur.

19 An example of a project of pure investment is given, if $S(t_n) = 0$, by the management of a loan at the rate r from the viewpoint of the lender. An example of a pure financing project is given, if $S(t_n) = 0$, by managing a loan at the rate k from the viewpoint of the borrower. For the management of loan with amortization, see Chapter 6.

$$\begin{cases} S_0 = a_0 \\ S_t = S_{t-1}(1+r) + a_t \quad if \;\; S_{t\text{-}1} < 0 \\ S_t = S_{t-1}(1+k) + a_t \quad if \;\; S_{t\text{-}1} \geq 0 \\ t = 1,...,n \end{cases} \tag{4.29}$$

Both for mixed and pure projects the value

$$S(k,r) = S_n = S_n(k,r) \tag{4.30}$$

can be >=< 0, constituting the final result of the project (from the viewpoint of the entrepreneur), given by the returns, which are also financial, the net of costs, which are also financial, at the final time instead of at the initial time (as seen for the present value), and on the basis of a complex and non-decomposable financial law, identified by (4.29), if the project is mixed and a non-reciprocal rate is applied, i.e. if r and k are both used for calculation and they are different.

We can now see the substantial limit of the present value criterion (see section 4.4.4) which is not useful for projects that interchange in time the role of investment taking money (if $S_t < 0$) and that of loan giving money (if $S_t > 0$) because we use only one evaluation rate whereas the rules of the money and financial market usually give rise to different return rates in the two cases.

The specification of the rate, when the pure or mixed feature of the project is specified, is needed because such a feature depends on the level of the investment or financing rate. In fact, every *mixed* project with $a_0 < 0$, when r increases and exceeds a given r-min, becomes a project of *pure investment* and the result does not depend on k; in the same way every mixed project with $a_0 > 0$, when k increases beyond a given k-min, becomes a project *of pure financing* and the result does not depend on r.

The property can be proved immediately observing that, if $a_0 < 0$ and thus the interest is initially accrued at the rate r, there exists a proper r-min, such that, $\forall h$ = 1,...,n-1, the absolute value of the decrement S_h - $S_{h\text{-}1}$ exceeds every a_h >0. Thus the balances, initially negative, remain negative for the whole time-length before the last transaction that gives the final result and the project results of a pure investment. Analogously, if $a_0 > 0$ and then the interest is initially accrued at the rate k, there exists a proper k-min such that $\forall h$ = 1,...,n-1 the increment S_h - $S_{h\text{-}1}$ exceeds the absolute value of each $a_h \leq 0$. Thus the balances, initially positive, remain positive for the whole time length before the last transaction that gives the final result and the project results of a pure financing.

Let us consider the geometric approach on the Cartesian plane. Restricting oneself to the *mixed projects* on the basis of the given rates (then with $r<r$-min if $a_0 < 0$, with $k<k$-min if $a_0 > 0$), while (4.30) generalizes the concept of present value of a project, the generalization of the internal rate is obtained choosing among the level curves of the surface (4.30), defined on the quadrant $k \geq 0$, $r \geq 0$ of the plane *Okr*, the one corresponding to the parameter = 0 and then to a nil final result. Analogously the IRR is solution $\hat{i}$ of the equation $V(\hat{i}) = 0$. Therefore the equation

$$S(k,r) = 0 \tag{4.31}$$

implicitly defines on the aforementioned quadrant a continuous and increasing curve, that is called *final fairness curve* (or more briefly, *fairness curve*, or also *curve of equilibrium*) of the project, generally asymptotic to the straight line $r = r$-min if $a_0 < 0$ or to the straight line $k = k$-min if $a_0 > 0$. The explicit forms of (4.31), to be considered alternatively, can be written as

$$\begin{cases} r = r_0(k) \\ k = k_0(r) \end{cases} \tag{4.32}$$

where r_0 and k_0 are the functional operators, one inverse of the other, which realize the final fairness of the project. Any pair $(\hat{k}, \hat{r})$ satisfying (4.31) and then, because of (4.32), such that

$$\hat{r} = r_0(\hat{k}) \text{ or } \hat{k} = k_0(\hat{r}) \tag{4.33}$$

identifies a point on the fairness curve and then has the following meaning: given the cash-flow of the mixed project, if in the investment stages of the project, interests (positive for the entrepreneur) are debited to it at the rate $\hat{r}$, such that the project is in equilibrium (i.e. gives rise to a null final balance) we have to credit interest (negative for the entrepreneur) at the rate $\hat{k}$ during the financing stages of the project. And the inverse holds. The points on curve (4.31), with coordinates of type $(\hat{k}, \hat{r})$, are financially equivalent in the sense that they all assure a zero final result

THEOREM.– The generalization of IRR implies a connection between *internal* allowed and charged interest rates, i.e. needed to maintain the equilibrium of the project, with necessarily concordant variations rate.

Proof

Given that it is not restrictive to suppose that the function *S(k,r)* has continuous first order partial derivatives, we have

$$\frac{\partial S}{\partial k} > 0 \quad ; \quad \frac{\partial S}{\partial r} < 0 \tag{4.34}$$

This is because according to (4.29), on curve (4.31) if *k* increases, fixing *r*, at the end of the process, *S* increases; while if *r* increases, fixing *k*, *S* decreases. Given that the derivative of the function $r = r_0(k)$ exists and is continuous, explicit equation of the curve *S*=0. Moreover, $\frac{dr_0(k)}{dk} = \frac{dr}{dk} = -\frac{\partial S}{\partial k} / \frac{\partial S}{\partial r} > 0$ holds. Analogous conclusion for the inverse function $k = k_0(r)$ with derivative $\frac{dk_0(r)}{dr} = \frac{dk}{dr} = -\frac{\partial S}{\partial r} / \frac{\partial S}{\partial k} > 0$.

On the contrary, if $a_0 < 0$ and *r* = *r*-min or $a_0 > 0$ and *k* = *k*-min, the project becomes *pure* and the *fairness curve* becomes a parallel to the *r* = 0 or *k* = 0 axis respectively. In such a case it is enough to consider only one of the *external rates*, *r** of cost or *k** of return; then the problem is led back to the one-dimensional case and to the criteria as given in sections 4.4.4 and 4.4.5.

Summarizing the geometric point of view, on varying the supplies, the fairness curves (4.31), locus of the points (*k*,*r*), cover: a) if $a_0 < 0$, a *mixed* region defined by (*k*>0, 0<*r*<*r*-min) and in the top a region as pure investment defined by (*k*>0, *r*≥*r*-min); b) if $a_0 > 0$, a *mixed* region defined by (0<*k*<*k*-min, *r*>0) and in the right side a region as *pure financing* defined by (*k*≥*k*-min, *r*>0) (see Figure 4.3).

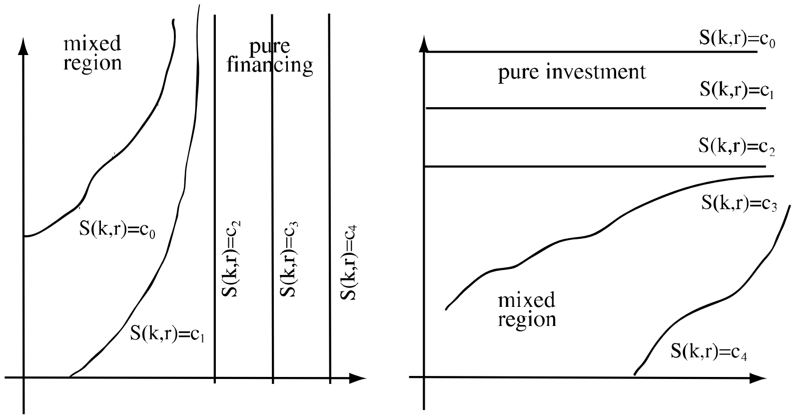

Figure 4.3a and b. *Fairness curves*

4.4.7. *Decisional criteria on mixed projects*

Let us now extend the decisional and choice criteria for mixed project as seen with the one rate approach in sections 4.4.4 and 4.4.5. In addition, we observe that it is enough to consider here decisional criteria: indeed, substitutive operations can be led back to a mixed project and we have shown in section 4.4.5 the way in which decision on them is equivalent to choices in alternative between projects of investment or financing. The decisional criterion discussed later has a geometric interpretation in Figure 4.4 with reference to the plane *Okr*.

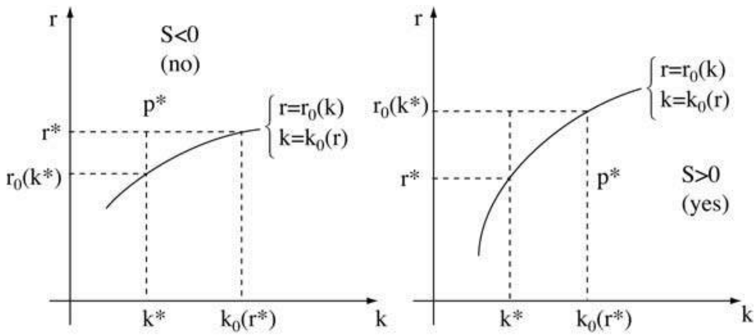

Figure 4.4a and b. *Plot of the decisional criterion*

When we are interested in rates that generate a mixed project (i.e. when there is plurality in the sign of the periodic balances sequence), from the viewpoint of the entrepreneur, the broker between the project and the financial market, the following decisional criteria apply.

a) *The final result (FR) criterion, which extends the present values (PV) criterion*

The decision depends on the sign of the final balance evaluated at the external cost rates *r** and returns *k**, already defined, on the capital market. Therefore, the criterion can be formulated as follows.

If

$$S(k^*,r^*) > 0 \qquad (4.35)$$

the mixed project is convenient *for the firm; if*

$$S(k^*,r^*) < 0 \qquad (4.35')$$

the mixed project is inconvenient *for the firm; if*

$$S(k^*,r^*) = 0 \tag{4.35''}$$

the mixed project is indifferent *for the firm.*

Proof

It is obvious that (4.35), (4.35') and (4.35") generalize the criterion of present value described in section 4.4.4, which uses the relations $V(x^*)>0$, $V(x^*)<0$ and $V(x^*)=0$. To prove the FR criterion with a direct argument, it is enough to observe that, from the viewpoint of the acting firm, the amounts of the cash-flow towards the market regarding the examined project are exactly the opposite of those of the cash-flow towards the project. In other words, the firm takes the amounts from the market with charged interest rate k^* and invests them in the project at the allowed rate $\hat{r}$, and, in addition, takes the amounts from the project with charged interest rate $\hat{k}$ and invests them in the market at the allowed rate r^*. If the final effect of such a transaction is a spread $S(k^*,r^*)>0$, then the mixed project is convenient, otherwise it is not.

b) *With the return and cost rates (RCR) criterion, which extends the internal rate (IRR) criterion*

In such a criterion (which, in contrast to the IRR criterion, can always be applied) it is necessary to consider four types of rates already indicated, i.e. the pair (k^*,r^*) of return and cost rates on the external market and the infinite pairs $(k_0(r),r_0(k))$, coordinates of points on the well established *fairness curve* which replaces the IRR, with varying $\hat{r}$ = *investment rate in the project* and $\hat{k}$ = *financing rate from the project*, from the viewpoint of the firm. The *RCR criterion* can then be formulated as follows.

Given a mixed project $\mathcal{P}$ *(at the considered rate) with a fairness curve of equation* (4.31) *with explicit form:* $r = r_0(k)$ *(i.e.:* $k = k_0(r)$*), using (from the viewpoint of the acting firm):*

– k^* = *external allowed return rate on the market;*

– r^* = *external charged cost rate on the market*[20]*;*

20 The original formulation of the TRM criterion, more limited than the one shown here, introduces only one external rate called *cost rate of the capital*, then assuming $k^*=r^*$ and considering the point P^* bounded on the bisector $r=k$. Thus, only the intersection points are meaningful of the fairness curve with the bisector $r=k$, which have coordinates equal to each other and to the IRR of the project, which are non-operative in the case of absence or plurality of solutions, leading back to the inconveniences of the IRR criterion. We consider more advantageous a more general schematization to complete the innovative contribution of this

– $k_0(r^*)$ = charged internal financing rate from the project, corresponding to the internal investment rate $\hat{r} = r^*$;

– $r_0(k^*)$ = allowed internal investment rate in the project, corresponding to the internal financing rate $\hat{k} = k^*$; in the hypothesis that the firm has access without limitation to the financing market at the rate r^* and to the investment market at the rate k^* (constant rates for the whole length of the mixed project), thus;

– if and only if the point $P^* = (k^*,r^*)$ is below the fairness curve, where

$$r_0(k^*) > r^* ; \quad k_0(r^*) < k^* \tag{4.36}$$

(the inequalities are either both true or both false due to the increasing behavior of the fairness curve), the project $\mathcal{P}$ is convenient *for the firm;*

– with the same hypothesis and positions, if and only if the point $P^* = (k^*,r^*)$ is above the fairness curve (4.31), where

$$r_0(k^*) < r^* ; \quad k_0(r^*) > k^* \tag{4.36'}$$

(the inequalities are either both true or both false), the project $\mathcal{P}$ is inconvenient *for the firm;*

– with the same hypothesis and positions, if and only if the point $P^* = (k^*, r^*)$ is on the fairness curve (4.31), where

$$r_0(k^*) = r^*; \quad k_0(r^*) = k^* \tag{4.36''}$$

(the equalities are either both true or both false), the project $\mathcal{P}$ is indifferent *for the firm.*

Proof

(4.36") follows from the definition of fairness curve. In addition, we can verify the equivalence between (4.35) and (4.36), between (4.35') and (4.36'), and between (4.35") and (4.36"), from which the *RCR criterion* follows.

To prove the validity of the RCR criterion with a direct argument, we firstly observe that a project is convenient or inconvenient if it gives rise to a property variation that increases the return rate and/or decreases the cost rate or it gives rise to

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scheme, also due to the fact that in the capital market it is used to work with non-reciprocal rates.

a property variation with opposite effects. Besides, let us indicate by S_t the balance with accumulated interest until time t, as defined in (4.29).

So if, when $S_t < 0$, the firm has invested in the mixed project $\mathcal{P}$ at the rate r^* equal to the cost rate on the market (then without spread), then in equilibrium condition the financing from $\mathcal{P}$, when $S_t > 0$, is ruled by the rate $k_0(r^*)$. Therefore:

– if the second inequality of (4.36) holds, investing the profit from such financing on the market at the rate $k^* > k_0(r^*)$, the firm has a positive spread and $\mathcal{P}$ is *convenient*;

– if the second inequality of (4.36') holds, investing the profit from such financing on the market at the rate $k^* < k_0(r^*)$, the firm has a negative spread and $\mathcal{P}$ is *inconvenient*;

– if the second equality of (4.36") holds, investing the profit from such financing on the market at the rate $k^* = k_0(r^*)$, the firm has zero spread and $\mathcal{P}$ is *indifferent.*

Otherwise if, when $S_t > 0$, the firm is financed from the mixed project $\mathcal{P}$ at the rate k^* equal to the return rate on the market (then without spread), then in equilibrium condition the investment in $\mathcal{P}$, when $S_t < 0$, is ruled by the rate $r_0(k^*)$. Therefore

– if the first inequality of (4.36) holds, taking money on the market at the rate $r^* < r_0(k^*)$, the firm has a positive spread and $\mathcal{P}$ is *convenient*;

– if the first inequality of (4.36') holds, taking money on the market at the rate $r^* > r_0(k^*)$, the firm has a negative spread and $\mathcal{P}$ is *inconvenient*;

– if the first inequality of (4.36") holds, taking money on the market at the rate $r^* = r_0(k^*)$, the firm has zero spread and $\mathcal{P}$ is *indifferent.*

EXAMPLE 4.4.– Regarding decisions on mixed projects, we use the classical example of the *oil pump project* shown by *Lorie and Savage*[21] to be a typical substitutive operation in the industrial field. Let us suppose that from an oil well, containing crude valued US$20,000 (we are using low numbers for sake of brevity: it would be enough to assume as unit a suitable power of 10), oil is being extracting at time 0 with a pump system that enables the completion of the extraction in 2 years, and there will be a gross profit of US$10,000 at the end of the 1st year and the same at the end of the 2nd year. It is then necessary to evaluate at time 0 the convenience of the installation of a more efficient pump, with a substitution cost of US$1,600, which enables the extraction to be completed within one year, with a profit of US$20,000 before the end of the 1st year.

21 See Lorie and Savage (1955) (see footnote 16).

Not considering previous flows, which have no influence here, the project $\mathcal{A}$ (= old pump) implies the following supply sequence:

$$\mathcal{A} : (1; +10{,}000) \cup (2; +10{,}000)$$

while the project $\mathcal{B}$ (= new pump) implies the following supplies:

$$\mathcal{B} : (0; -1{,}600) \cup (1; +20{,}000).$$

Note that $\mathcal{A}$ does not have IRR and it is convenient with any evaluation rate, while $\mathcal{B}$ is a simple project with IRR = 1,150%. However, here, being a substitutive operation, we are interested in the difference project given by

$$\mathcal{A} - \mathcal{B}: (0, -1{,}600) \cup (1, +10{,}000) \cup (2, -10{,}000)$$

Now $\mathcal{B} - \mathcal{A}$ is a mixed project that starts with a payment and becomes a pure investment project only if $r \geq r$-min = 525%.

The final fairness curve $r = r_0(k)$ has equation

$$r = 5.25 - 6.25/(1 + k) \tag{4.37}$$

an explicit form of $S = 0$, which in this case is written as follows:

$$[-1{,}600\,(1 + r) + 10{,}000]\,(1 + k) - 10{,}000 = 0$$

On the basis of the original criterion TRM (see footnote 20), the intersections of (4.37) with the bisector $r=k$ correspond to the following IRR values:

$$k^* = r^* = 0.25 = 25\% \;;\quad k^* = r^* = 4 = 400\%$$

(non-operative IRR because we have obtained more solutions). Therefore, the substitution of the pump, i.e. the change from $\mathcal{A}$ to $\mathcal{B}$ is convenient if and only if the market rate, reciprocal for investments and financings, chosen for the evaluation is between 25% and 400%. Indeed, the fairness curve (4.37) has the concavity downwards and, intersecting the 1st bisector in points P_1 = (0.25;0.25) and P_2 = (4;4), all points P^* of such a bisector between P_1 and P_2, with $0.25 < k^* = r^* < 4$, are such that $S(P^*) > 0$. So, with such external rates the substitution is convenient. Instead, if $k^* = r^*$ is > 4 or < 0.25, the substitution is inconvenient.

On the basis of the version of TRM introduced here, which considers market rates k^* and r^* to be different, it is necessary to evaluate the pair (k^*,r^*) to adopt and accept the substitution if and only if the point $P^* = (k^*,r^*)$ is below the curve (4.37).

Observe that the oil company, if it performs the substitution, will have to obtain at time 0 a loan of US$1,600 at the cost rate r^* and will have to invest at time 1 the higher profit of US$10,000 at the return rate k^*.

4.5. Appendix: outline on numerical methods for the solution of equations

4.5.1. *General aspects*

As seen in this chapter and as will be seen in Chapter 5 in the particular case of *annuity flows*, the congruity relations between the flows of a financial operation and their capital values at a given time are often considered in financial mathematics under different hypotheses on the adopted financial laws. However, such a relation can be thought of as equations where the unknown is the length or the rate, and all other quantities are given. The duration is seldom considered unknown indeed, while the rate is often considered like this. Then there is the classic problem of the *calculation of the IRR of a financial operation O*. We saw the importance of this in the previous sections, but we did not consider its calculation.

The solution of such a problem is not simple and sometimes it is impossible from the algebraic viewpoint, when the equation on the rate is not simple enough to give a solution in closed form. It is then necessary to apply numerical methods that give approximate solutions, unless iterative methods are applied to obtain the numerically exact solution in the desired number of decimals. The field of application of such methods is much greater than the calculation of IRR. We then consider it opportune to give a brief insight into the theoretical and applicative aspects of more suitable numerical methods even if the software available on PCs enables an easy evaluation of equation roots, in particular the IRR of an operation *O*.

On the choice of the calculation methods for IRR, it is necessary to consider that due to the versatility and popularity of PCs, and also of the pocket calculator, many of the methods used in the past are now obsolete. We will consider few classic methods, favoring the iterative ones.

4.5.2. *The linear interpolation method*

Given a function *f(x)*, which is continuous and monotonic in an assigned interval, and given a value *k*, it is necessary to find the value $\tilde{x}$ such that $f(\tilde{x})=k$, i.e. the root of the equation:

$$g(x) := f(x) - k = 0 \tag{4.38}$$

The linear interpolation is done starting from the values x_1 and x_2, which are close enough between them and to the root[22], and such that $f(x_1) > k$, $f(x_2) < k$ [23] as well. Then an approximate estimation of $\tilde{x}$, which we will indicate by x_0, is obtained from the abscissa of the intersection with $y = k$ of the secant line to the graph of $y=f(x)$ in the points with abscissas x_1 and x_2. Indeed, x_0 is also the root of the secant the graph of $g(x)$ in the points with abscissas x_1 and x_2. This easily results in

$$x_0 = x_1 + \frac{k - f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1) \tag{4.39}$$

22 The search for x_1 and x_2 "by chance" is not easy and not very scientific. In the financial problems in which we are interested, a starting approximate value $\hat{x}$ of the solution $\tilde{x}$, in order to find by small steps such values x_1 and x_2 , can be easily obtained on the basis of linear hypothesis, using the arithmetic mean and the SDI law. In particular we consider an *O* satisfying the condition that the arithmetic mean maturity of the payments precedes the time of first income; we know that it is enough for *O* to be an investment in the broad sense and then to have an operative IRR. Let us denote by τ_i the arithmetic mean of incomes, with τ_e that of payments, with *E* the sum of payments and with *I* the sum of incomes. Thus, being $\tau_e < \tau_i$, the IRR approximated $\hat{x}$ is found from: $E[1+(\tau_i-\tau_e)\hat{x}] = I$. In the particular case of simple investment with payment V_0 in 0 and incomes R_h in *h*, the arithmetic mean maturity of payments is then $\tau_e = 0$, that of incomes is $\tau_i = \sum_h hR_h / \sum_h R_h$ and $\hat{x}$ follows from: $V_0(1 + \tau_i \hat{x}) = \sum_h R_h$. We have such a situation in the amortization of the debt V_0 with installments R_h (see Chapters 5 and 6).

23 Then $x_1 > x_2$ if *f* increases in the interval, and $x_1 < x_2$ if *f* decreases. If *f(x)* is the initial value of an annuity at rate *x*, then *f* is decreasing and convex, so $\tilde{x} < x_0$ (see Chapter 5).

Proof

The secant defined before is the straight line through $P_1 \equiv [x_1, f(x_1)]$ and $P_2 \equiv [x_2, f(x_2)]$, with equation

$$\frac{y - f(x_1)}{f(x_2) - f(x_1)} = \frac{x - x_1}{x_2 - x_1} \tag{4.39'}$$

and making a system with $y=k$, the solution $x=x_0$ expressed by (4.39) is obtained.

The solution x_0 is an approximation *by excess*, i.e. greater than the exact value, if $f(x)$ (and then $g(x)$) is also decreasing and convex (i.e. with upwards concavity)[24] or increasing and concave (i.e. with downwards concavity); instead x_0 is an approximation *by defect*, i.e. smaller than the exact value, if $f(x)$ (and then $g(x)$) is decreasing and concave or increasing and convex.

The linear interpolation procedure can be iterated using in the procedure the found approximated root, i.e., using the initial positions, acting analogously on the interval (x_0, x_1) or alternatively (x_0, x_2), thus satisfying the condition of sign discordance between $f(x_i)-k$, i = 1 or 2, and $f(x_0)-k$. The estimation $x_0^{(2)}$ of the root is then obtained. Proceeding analogously again we obtain a sequence $x_0^{(i)}$, i = 2, 3,... converging to the root $\tilde{x}$, i.e. such that $f(x_0^{(i)})$ converges to k; this procedure gives the *secant method* (see below)

Exercise 4.6

To buy a shed to be used in an industrial company for the price of €170,000, the entrepreneur sells stocks earning €22,000 and for the remaining part he enters into a loan to repay (for the amortization procedure see Chapter 6) with 10 annual delayed installments $R_h = R+hD$, (h = 1,...,10), where R = 17,030, $D = 0.04R$; then R_1= 17,711.20; R_2 = 18,392.40;.....; R_{10} = 23,842.00. Find the loan rate.

A. The loaned amount is S = 170,000 – 22,000 = €148,000; the length is n = 10; the installments are given. To find the rate we apply the linear interpolation method on the function $f(x)$, related to the equivalence (6.2) specified in section 6.2

$$S = f(x) \coloneqq \sum_{h=1}^{10} R_h (1+x)^{-h}$$

We start from a rough rate $\hat{x}$ according to footnote 22. This results in

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24 We find this case in the search for IRR in an amortization operation for a capital given as loan, with given cash-flow and maturity of installments (see Chapter 6), solving an equation of type $V(i) = 0$, with $V(i)$ decreasing and with upwards concavity.

$$\tau_i = \sum_h h(R+hD)/\sum_h (R+hD)$$ = (55+0.04·385)/(10+0.04·55) = 5.77; V_0 $(1+\tau_i\,\hat{x}) = \sum_h R_h$ becomes: 148,000 (1+ 5.77 $\hat{x}$) = 207.766

then: $\hat{x} = 0.06999$. We obtain: g(0.06999) = -4,724.11 and with $g(x)$ decreasing the root is a value < 7%. Acting on $f(x)$, with decreasing x we easily obtain using Excel

f(0.0675) = 145,037.74

f(0.0650) = 146,832.06

f(0.0625) = 148,659.58

The solution $\tilde{x}$ is evidently between 6.50% and 6.25%. Applying (4.39) we obtain:

$$x_0 = 0.0625 + \frac{148,000.00 - 148,659.58}{146,832.06 - 148,659.58}0.025 = 0.06340$$

In accordance with the fact that x_0 is approximation by excess, f(0.0634) = 147,996.13 < 148,000 follows. Interpolating on the interval (0.0625; 0.0634) (2nd step of the “secant method”, of which the starting interpolation is the 1st) we obtain:

$$x_{00} = 0.0625 + \frac{148,000.00 - 148,659.58}{147,996.13 - 148,659.58}(0.0634 - 0.062) = 0.063395$$

The process can be iterated again if we require many exact decimal digits; otherwise the rate 6.3395% is a satisfactory estimation of the loan rate, giving rise to a relative spread of less than 10^{-4}.

Financial application with numerical solution using the linear interpolation methods has been developed in section 4.4.4, Example 4.3.

4.5.3. *Dichotomic method (or for successive divisions)*

This is a procedure with slow convergence, which can be applied, due to its simplicity, if we have a calculator.

In order to solve equation (4.38), assuming $f(x)$ to be continuous and monotonic, by assumption we know that the searched root is in an interval with known extreme a, b, with $a<b$, considering the case: $f(a) > k$, $f(b) < k$.

Let us define $x_0 = (a+b)/2$, $d = |a–b|$ and then, with recursive process,

$$x_h = x_{h-1} \pm \frac{d}{2^h} \quad ; \quad h = 1,\ 2,\ ... \tag{4.40}$$

using in (4.40) the sign + or - according to, respectively, $f(x_{h-1}) > k$ or $f(x_{h-1}) < k$. In such a case we obtain a sequence $x_0, x_1,\ldots,x_h,\ldots$ approximating a root $\tilde{x}$ of (4.38), in the sense that the pairs of consecutive values x_{h-1}, x_h are extremes of numeric interval containing $\tilde{x}$ and of amplitude $d/2^h$ geometrically decreasing with ratio 1/2.

4.5.4. *Secants and tangents method*

Given equation (4.38) with *f(x)* continuous and differentiable, considering the case that *f(x)* is decreasing and has upwards concavity (the changes are obvious for the other cases), it is necessary to find the abscissa $\tilde{x}$ of the point X intersection of the graphs $y = f(x)$ and $y = k$ (see Figure 4.5). The procedure on the secants being considered here is a particular case of that shown in section 4.5.2.

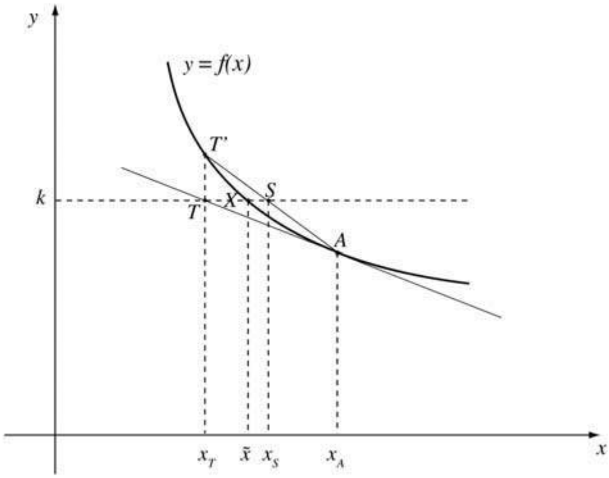


Figure 4.5. *Secants and tangents method*

Initial step

Starting from point *A* of the *f(x)* graph with abscissa $x_A > \tilde{x}$ (where $f(x_A) < k$), possibly already approximated to $\tilde{x}$ on the basis of preliminary information (see footnote 22), the *tangent* to the curve in *A* is analytically found, whose equation is

$$y - f(x_A) = f'(x_A) - (x - x_A) \tag{4.41}$$

and by using $y=k$ we obtain the abscissa x_T of the point T intersection of the tangent with the line $y=k$:

$$x_T = x_A + (k - f(x_A))/f'(x_A) \tag{4.42}$$

Due to a well known property of the upwards concave function, the inequalities $x_T < \tilde{x} < x_A$ hold. Then (4.42) is an estimate of $\hat{x}$ approximated by defect.

Furthermore, we obtain the equation of the *secant* to the curve through A and the point T' of the graph with abscissa of T, obtaining

$$\frac{y - f(x_A)}{f(x_T) - f(x_A)} = \frac{x - x_A}{x_T - x_A} \tag{4.43}$$

and using $y=k$ we find from (4.43) that x_S is an approximation that is better than x_A. We then find the numeric interval, with extremes x_T , x_S, which includes the searched solution $\tilde{x}$.

Next step

The procedure can be iterated, starting from x_S, instead of x_A, obtaining an approximated interval, which is contained in the previous intervals and has decreasing amplitude converging to 0, thus obtaining a very good estimate of $\tilde{x}$.

4.5.5. *Classical iteration method*

Let us give a brief insight on a widely-applied method – *classical iteration* – which requires the availability of a personal computer (or a programmable calculator). Let us consider an equation, written in the form $x = f(x)$, where $f(x)$ exists continuously in an interval containing the root α, for which the approximate value x_0 is known. If the equation is given in the form: $g(x) = g_0$, it is enough to use: $f(x) = x + g(x) - g_0$ or, if $g_0 \neq 0$, to use: $f(x) = x\, g(x)/g_0$. The method consists of finding the following sequence, starting from x_0:

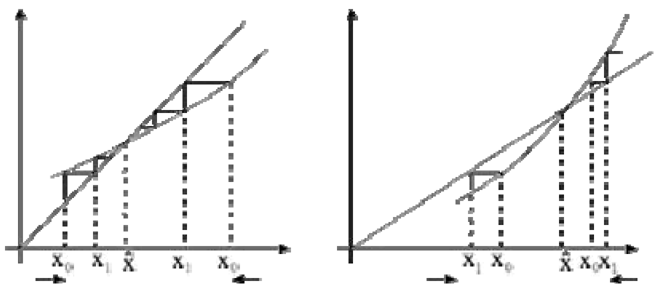
$$x_h = f(x_{h-1}); \quad (h = 1, 2, 3, \ldots) \tag{4.44}$$

because if $\{x_h\}$ converges, its limit is necessarily given by the root of the given equation.

The validity of the procedure arises from the following theorem.

THEOREM.– *If there exists a root* α *of the equation* $x = f(x)$ *– where* $f(x)$ *is defined, continuous and differentiable in the interval J containing* α *and* x_0 *is a value of first approximation of* α *– and moreover if, for a given prefixed constant H satisfying:* $0 < H < 1$, *we obtain* $\forall x \in J$: $|f'(x)| \le H$, *then the sequence* $\{x_h\}$ *converges and its limit is the root* α. *If, instead, in J we obtain:* $|f'(x)| > 1$, *then the sequence does not converge to* α[25].

Figures 4.6a and b geometrically show the convergence and divergence cases of the iteration method.



(convergence to $\hat{x}$) (divergence from $\hat{x}$)

Figure 4.6a and b. *Convergence and divergence cases of the iteration method*

Referring to the graphs of $y = x$ and $y = f(x)$, it is geometrically obvious that the convergence of the procedure is faster the closer the value of $|f\,'(\alpha)|$ is to zero. Therefore, the following transformation is used to accelerate the convergence or to make it possible when it is not on f, i.e. if $|f\,'(\alpha)| > 1$. Using

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25 Let us give a brief proof of the theorem. If $|f'(x)| \le H$ with $0<H<1$, due to the Cavalieri-Lagrange theorem we can write, given that by definition $\tilde{x} = f(\tilde{x})$, recalling (4.44) and introducing $\bar{x}_0$ between $\tilde{x}$ and x_0):

$$\left|\tilde{x} - x_1\right| = \left|f(\tilde{x}) - f(x_0)\right| = \left|f'(\bar{x}_0)\right| \left|\tilde{x} - x_0\right| \le \alpha\left|\tilde{x} - x_0\right| < \left|\tilde{x} - x_0\right|$$

and analogously, for $h = 2,3,\ldots$

$$\left|\tilde{x} - x_h\right| = \left|f(\tilde{x}) - f(x_{h-1})\right| = \left|f'(\bar{x}_{h-1})\right| \left|\tilde{x} - x_{h-1}\right| \le \alpha\left|\tilde{x} - x_{h-1}\right| \le \alpha^h\left|\tilde{x} - x_0\right|$$

Because of the formula: $\lim\limits_{h\to\infty} \alpha^h = 0$, and of a comparison theorem,

$$0 \le \lim_{h\to\infty}\left|\tilde{x} - x_h\right| = \left|\tilde{x} - x_0\right| \lim_{h\to\infty} \alpha^h = 0 \qquad \text{that is} \qquad \lim_{h\to\infty} x_h = \tilde{x}$$

follows, and then the thesis. Again for a comparison theorem, if $|f'(x)|>1$ holds in a neighborhood of $\tilde{x}$, a diverging geometric sequence is minorant of $\{x_h - x_0\}$, therefore this sequence is also diverging and its values deviate from the solution $\tilde{x}$.

$$m = f'(\alpha) \quad ; \quad g(x) = [f(x) - m\,x]/(1 - m) \tag{4.45}$$

$g'(\alpha) = 0$ follows, and it is easy to verify that the equation $x = g(x)$, to which the iteration method can be applied in the best conditions, is equivalent to $x = f(x)$, so it has the same roots. Obviously m cannot at first be found in an exact way because it is unknown α; however, an approximate value almost satisfies the condition and then the method can be applied in order to obtain a quick convergence.

Financial applications using the classical iteration method for numerical solutions have been developed in section 4.4.4, Example 4.3.